MATH 311

Topics in Applied Mathematics I

**Lecture 6:
Diagonal matrices.
Inverse matrix.**

## Diagonal matrices

If $A = (a_{ij})$ is a square matrix, then the entries $a_{ii}$ are called **diagonal entries**. A square matrix is called **diagonal** if all non-diagonal entries are zeros.

*Example.* $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, denoted $\mathrm{diag}(7, 1, 2)$.

Let $A = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, $B = \mathrm{diag}(t_1, t_2, \ldots, t_n)$.

Then $A + B = \mathrm{diag}(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$,

$rA = \mathrm{diag}(rs_1, rs_2, \ldots, rs_n)$.

*Example.*

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

**Theorem** Let $A = \operatorname{diag}(s_1, s_2, \ldots, s_n)$,
$B = \operatorname{diag}(t_1, t_2, \ldots, t_n)$.

Then $A + B = \operatorname{diag}(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$,
$$rA = \operatorname{diag}(rs_1, rs_2, \ldots, rs_n).$$
$$AB = \operatorname{diag}(s_1 t_1, s_2 t_2, \ldots, s_n t_n).$$

In particular, diagonal matrices always commute
(i.e., $AB = BA$).

*Example.*

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 7a_{11} & 7a_{12} & 7a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{pmatrix}$$

**Theorem** Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_m)$ and $A$ be an $m \times n$ matrix. Then the matrix $DA$ is obtained from $A$ by multiplying the $i$th row by $d_i$ for $i = 1, 2, \ldots, m$:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \implies DA = \begin{pmatrix} d_1\mathbf{v}_1 \\ d_2\mathbf{v}_2 \\ \vdots \\ d_m\mathbf{v}_m \end{pmatrix}$$

*Example.*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 7a_{11} & a_{12} & 2a_{13} \\ 7a_{21} & a_{22} & 2a_{23} \\ 7a_{31} & a_{32} & 2a_{33} \end{pmatrix}$$

**Theorem** Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and $A$ be an $m \times n$ matrix. Then the matrix $AD$ is obtained from $A$ by multiplying the $i$th column by $d_i$ for $i = 1, 2, \ldots, n$:

$$A = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$$
$$\implies AD = (d_1\mathbf{w}_1, d_2\mathbf{w}_2, \ldots, d_n\mathbf{w}_n)$$

## Identity matrix

*Definition.* The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The $n \times n$ identity matrix is denoted $I_n$ or simply $I$.

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, $I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$

**Theorem.** Let $A$ be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

## Inverse matrix

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. We can **add**, **subtract**, and **multiply** elements of $\mathcal{M}_n(\mathbb{R})$. What about **division**?

*Definition.* Let $A \in \mathcal{M}_n(\mathbb{R})$. Suppose there exists an $n \times n$ matrix $B$ such that

$$AB = BA = I_n.$$

Then the matrix $A$ is called **invertible** and $B$ is called the **inverse** of $A$ (denoted $A^{-1}$).

A non-invertible square matrix is called **singular**.

$$\boxed{AA^{-1} = A^{-1}A = I}$$

## Examples

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \ B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \ \ C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $A^{-1} = B$, $B^{-1} = A$, and $C^{-1} = C$.

## Basic properties of inverse matrices

• If $B = A^{-1}$ then $A = B^{-1}$. In other words, if $A$ is invertible, so is $A^{-1}$, and $A = (A^{-1})^{-1}$.

• The inverse matrix (if it exists) is unique. Moreover, if $AB = CA = I$ for some $n \times n$ matrices $B$ and $C$, then $B = C = A^{-1}$.

Indeed, $B = IB = (CA)B = C(AB) = CI = C$.

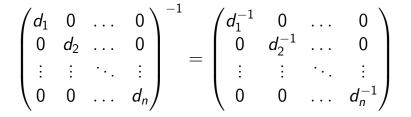• If $n \times n$ matrices $A$ and $B$ are invertible, so is $AB$, and $(AB)^{-1} = B^{-1}A^{-1}$.

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$
$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$

• Similarly, $(A_1 A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1} A_1^{-1}$.

## Inverting diagonal matrices

**Theorem** A diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If $D$ is invertible then $D^{-1} = \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

$$
\begin{pmatrix}
d_1 & 0 & \ldots & 0 \\
0 & d_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & d_n
\end{pmatrix}^{-1}
=
\begin{pmatrix}
d_1^{-1} & 0 & \ldots & 0 \\
0 & d_2^{-1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & d_n^{-1}
\end{pmatrix}
$$

## Inverting diagonal matrices

**Theorem** A diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If $D$ is invertible then $D^{-1} = \operatorname{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

*Proof:* If all $d_i \neq 0$ then, clearly,

$$\operatorname{diag}(d_1, \ldots, d_n) \operatorname{diag}(d_1^{-1}, \ldots, d_n^{-1}) = \operatorname{diag}(1, \ldots, 1) = I,$$
$$\operatorname{diag}(d_1^{-1}, \ldots, d_n^{-1}) \operatorname{diag}(d_1, \ldots, d_n) = \operatorname{diag}(1, \ldots, 1) = I.$$

Now suppose that $d_i = 0$ for some $i$. Then for any $n \times n$ matrix $B$ the $i$th row of the matrix $DB$ is a zero row. Hence $DB \neq I$ as $I$ has no zero rows.