MATH 311 Topics in Applied Mathematics I Lecture 21a: Similar matrices.

Change of basis for a linear operator

Let $L: V \to V$ be a linear operator on a vector space V.

Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$.



It follows that $UA\mathbf{x} = BU\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n \implies UA = BU$. Then $A = U^{-1}BU$ and $B = UAU^{-1}$.

Similarity of matrices

Definition. An $n \times n$ matrix B is said to be **similar** to an $n \times n$ matrix A if $B = S^{-1}AS$ for some nonsingular $n \times n$ matrix S.

Remark. Two $n \times n$ matrices are similar if and only if they represent the same linear operator on \mathbb{R}^n with respect to different bases.

Theorem Similarity is an equivalence relation, which means that (i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Corollary The set of $n \times n$ matrices is partitioned into disjoint subsets (called *similarity classes*) such that all matrices in the same subset are similar to each other while matrices from different subsets are never similar.

Theorem Similarity is an equivalence relation, i.e.,
(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Proof: (i)
$$A = I^{-1}AI$$
.
(ii) If $B = S^{-1}AS$ then $A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$
 $= S_1^{-1}BS_1$, where $S_1 = S^{-1}$.
(iii) If $A = S^{-1}BS$ and $B = T^{-1}CT$ then
 $A = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS) = (TS)^{-1}C(TS)$
 $= S_2^{-1}CS_2$, where $S_2 = TS$.

Theorem If A and B are similar matrices then they have the same (i) determinant, (ii) trace = the sum of diagonal entries, (iii) rank, and (iv) nullity.