MATH 311

Topics in Applied Mathematics I

Lecture 32:
Gradient, divergence, and curl.
Review of integral calculus.

Gradient, divergence, and curl

Gradient of a scalar field $f = f(x_1, x_2 \ldots, x_n)$ is

$$\operatorname{grad} f = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\Big).$$

Divergence of a vector field $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ is

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

Curl of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is

$$\operatorname{curl} \mathbf{F} = \Big(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3},\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1},\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\Big).$$

Informally, $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}.$

Del notation

Gradient, divergence, and curl can be denoted in a compact way using the del (a.k.a. nabla a.k.a. atled) "operator"

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right).$$

Namely, $\operatorname{grad} f = \nabla f$, $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$, $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$.

Theorem 1 $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$ wherever the vector field $\mathbf{F}$ is twice continuously differentiable.

Theorem 2 $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$ wherever the scalar field f is twice continuously differentiable.

In the del notation, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ and $\nabla \times (\nabla f) = \mathbf{0}$.
Note that $\nabla \cdot \nabla f = \Delta f$ (the Laplacian, also denoted $\nabla^2 f$).

Riemann sums and Riemann integral

Definition. A **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ with respect to a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

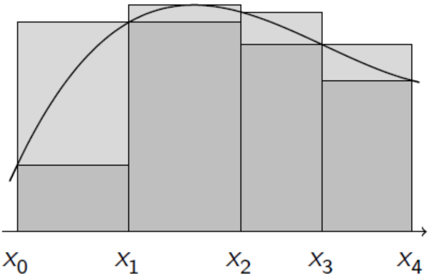
$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\,(x_j - x_{j-1}).$$

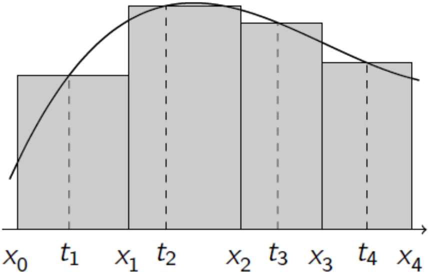
Remark. $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ if $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The norm of the partition P is $\|P\| = \max_{1 \le j \le n} |x_j - x_{j-1}|$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

If this is the case, then the function f is called **integrable** on $[a, b]$ and the limit $I(f)$ is called the **integral** of f over $[a, b]$, denoted $\int_a^b f(x)\,dx$.

Riemann sums and Darboux sums





Integration as a linear operation

Theorem 1 If functions f, g are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Theorem 2 If a function f is integrable on $[a, b]$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on $[a, b]$ and

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

More properties of integrals

Theorem If a function f is integrable on $[a, b]$ and $f([a, b]) \subset [A, B]$, then for each continuous function $g : [A, B] \to \mathbb{R}$ the composition $g \circ f$ is also integrable on $[a, b]$.

Theorem If functions f and g are integrable on $[a, b]$, then so is fg.

Theorem If a function f is integrable on $[a, b]$, then it is integrable on each subinterval $[c, d] \subset [a, b]$. Moreover, for any $c \in (a, b)$ we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Comparison theorems for integrals

Theorem 1 If functions f, g are integrable on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Theorem 2 If f is integrable on $[a, b]$ and $f(x) \ge 0$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx \ge 0$.

Theorem 3 If f is integrable on $[a, b]$, then the function $|f|$ is also integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Fundamental theorem of calculus

Theorem If a function f is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b],$$

is continuously differentiable on $[a, b]$. Moreover, $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem If a function F is differentiable on $[a, b]$ and the derivative F' is integrable on $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Problem. Evaluate $\displaystyle\int_0^1 \frac{x(x-3)}{(x-1)^2(x+2)}\,dx.$

To evaluate the integral, we need to decompose the rational function $R(x) = \frac{x(x-3)}{(x-1)^2(x+2)}$ into a sum of partial fractions:

$$
\begin{aligned}
R(x) &= \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2} \\
&= \frac{a(x-1)(x+2) + b(x+2) + c(x-1)^2}{(x-1)^2(x+2)} \\
&= \frac{(a+c)x^2 + (a+b-2c)x + (-2a+2b+c)}{(x-1)^2(x+2)}.
\end{aligned}
$$

$$
\begin{cases}
a + c = 1 \\
a + b - 2c = -3 \\
-2a + 2b + c = 0
\end{cases}
$$

Change of the variable in an integral

Theorem If ϕ is continuously differentiable on a closed interval $[a,b]$ and f is continuous on $\phi([a,b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\,\phi'(x)\,dx = \int_a^b f(\phi(x))\,d\phi(x).$$

Remarks. • The **Leibniz differential** $d\phi$ of the function ϕ is defined by $d\phi(x) = \phi'(x)\,dx = \frac{d\phi}{dx}\,dx$.

• It is possible that $\phi(a) \geq \phi(b)$. Hence we set

$$\int_c^d f(t)\,dt = -\int_d^c f(t)\,dt$$

if $c > d$. Also, we set the integral to be 0 if $c = d$.

• $t = \phi(x)$ is a proper change of the variable only if the function ϕ is strictly monotone. However the theorem holds even without this assumption.

Problem. Evaluate $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$.

To integrate this function, we introduce a new variable $u = 1 - x^2$:

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_0^{1/2} \frac{(1-x^2)'}{\sqrt{1-x^2}}\,dx$$

$$= -\frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,d(1-x^2) = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du$$

$$= \int_{3/4}^{1} \frac{1}{2\sqrt{u}}\,du = \sqrt{u}\,\Big|_{u=3/4}^{1} = 1 - \frac{\sqrt{3}}{2}.$$

Sets of measure zero

Definition. A subset E of the real line $\mathbb{R}$ is said to have **measure zero** if for any $\varepsilon > 0$ the set E can be covered by a sequence of open intervals $J_1, J_2, \dots$ such that $\sum_{n=1}^{\infty} |J_n| < \varepsilon$.

Examples. • Any set E that can be represented as a sequence $x_1, x_2, \dots$ (such sets are called **countable**) has measure zero.
Indeed, for any $\varepsilon > 0$, let

$$J_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, \, x_n + \frac{\varepsilon}{2^{n+1}} \right), \quad n = 1, 2, \dots$$

Then $E \subset J_1 \cup J_2 \cup \dots$ and $|J_n| = \varepsilon / 2^n$ for all $n \in \mathbb{N}$ so that $\sum_{n=1}^{\infty} |J_n| = \varepsilon$.

• The set $\mathbb{Q}$ of rational numbers has measure zero (since it is countable).

• Nondegenerate interval $[a, b]$ is not a set of measure zero.

Lebesgue's criterion for Riemann integrability

Definition. Suppose $P(x)$ is a property depending on $x \in S$, where $S \subset \mathbb{R}$. We say that $P(x)$ holds for **almost all** $x \in S$ (or **almost everywhere** on S) if the set $\{x \in S \mid P(x) \text{ does not hold}\}$ has measure zero.

Theorem A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on the interval $[a, b]$ if and only if f is bounded on $[a, b]$ and continuous almost everywhere on $[a, b]$.