

MATH 311

Topics in Applied Mathematics I

Lecture 34:

Line integrals.

Green's theorem.

Path

Definition. A **path** in \mathbb{R}^n is a continuous function $\mathbf{x} : [a, b] \rightarrow \mathbb{R}^n$.

Paths provide parametrizations for curves.

Length of the path \mathbf{x} is defined as

$L = \sup_P \sum_{j=1}^k \|\mathbf{x}(t_j) - \mathbf{x}(t_{j-1})\|$ over all partitions $P = \{t_0, t_1, \dots, t_k\}$ of the interval $[a, b]$.

Theorem The length of a smooth path

$\mathbf{x} : [a, b] \rightarrow \mathbb{R}^n$ is $\int_a^b \|\mathbf{x}'(t)\| dt$.

Arclength parameter: $s(t) = \int_a^t \|\mathbf{x}'(\tau)\| d\tau$.

Scalar line integral

Scalar line integral is an integral of a scalar function f over a path $\mathbf{x} : [a, b] \rightarrow \mathbb{R}^n$ of finite length relative to the arclength. It is defined as a limit of Riemann sums

$$\mathcal{S}(f, P, \tau_j) = \sum_{j=1}^k f(\mathbf{x}(\tau_j)) (s(t_j) - s(t_{j-1})),$$

where $P = \{t_0, t_1, \dots, t_k\}$ is a partition of $[a, b]$, $\tau_j \in [t_j, t_{j-1}]$ for $1 \leq j \leq k$, and s is the arclength parameter of the path \mathbf{x} .

Theorem Let $\mathbf{x} : [a, b] \rightarrow \mathbb{R}^n$ be a smooth path and f be a function defined on the image of this path. Then

$$\int_{\mathbf{x}} f ds = \int_a^b f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| dt.$$

ds is referred to as the arclength element.

Vector line integral

Vector line integral is an integral of a vector field over a smooth path. It is a scalar.

Definition. Let $\mathbf{x} : [a, b] \rightarrow \mathbb{R}^n$ be a smooth path and \mathbf{F} be a vector field defined on the image of this path. Then
$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt.$$

Alternatively, the integral of \mathbf{F} over \mathbf{x} can be represented as the integral of a **differential form**

$$\int_{\mathbf{x}} F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n,$$

where $\mathbf{F} = (F_1, F_2, \dots, F_n)$ and $dx_i = x_i'(t) dt$.

Applications of line integrals

- Mass of a wire

If f is the density on a wire C , then $\int_C f ds$ is the mass of C .

- Work of a force

If \mathbf{F} is a force field, then $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ is the work done by \mathbf{F} on a particle that moves along the path \mathbf{x} .

- Circulation of fluid

If \mathbf{F} is the velocity field of a planar fluid, then the circulation of the fluid across a closed curve C is $\oint_C \mathbf{F} \cdot d\mathbf{s}$.

- Flux of fluid

If \mathbf{F} is the velocity field of a planar fluid, then the flux of the fluid across a closed curve C is $\oint_C \mathbf{F} \cdot \mathbf{n} ds$, where \mathbf{n} is the outward unit normal vector to C .

Line integrals and reparametrization

Given a path $\mathbf{x} : [a, b] \rightarrow \mathbb{R}^n$, we say that another path $\mathbf{y} : [c, d] \rightarrow \mathbb{R}^n$ is a **reparametrization** of \mathbf{x} if there exists a continuous invertible function $u : [c, d] \rightarrow [a, b]$ such that $\mathbf{y}(t) = \mathbf{x}(u(t))$ for all $t \in [c, d]$.

The reparametrization may be orientation-preserving (when u is increasing) or orientation-reversing (when u is decreasing).

Theorem 1 Any scalar line integral is invariant under reparametrizations.

Theorem 2 Any vector line integral is invariant under orientation-preserving reparametrizations and changes its sign under orientation-reversing reparametrizations.

As a consequence, we can define the integral of a function over a simple curve and the integral of a vector field over a simple oriented curve.

Green's Theorem

Theorem Let $D \subset \mathbb{R}^2$ be a closed, bounded region with piecewise smooth boundary ∂D oriented so that D is on the left as one traverses ∂D . Then for any smooth vector field $\mathbf{F} = (M, N)$ on D ,

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

or, equivalently,

$$\oint_{\partial D} M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

Examples

Consider vector fields $\mathbf{F}(x, y) = (-y, 0)$,
 $\mathbf{G}(x, y) = (0, x)$, and $\mathbf{H}(x, y) = (y, x)$.

According to Green's Theorem,

$$\oint_{\partial D} -y \, dx = \iint_D 1 \, dx \, dy = \text{area}(D),$$

$$\oint_{\partial D} x \, dy = \iint_D 1 \, dx \, dy = \text{area}(D),$$

$$\oint_{\partial D} y \, dx + x \, dy = \iint_D 0 \, dx \, dy = 0.$$

Green's Theorem

Proof in the case $D = [0, 1] \times [0, 1]$ *and* $\mathbf{F} = (0, N)$:

$$\int_0^1 \frac{\partial N}{\partial x}(\xi, y) d\xi = N(1, y) - N(0, y)$$

for any $y \in [0, 1]$ due to the Fundamental Theorem of Calculus. Integrating this equality by y over $[0, 1]$, we obtain

$$\iint_D \frac{\partial N}{\partial x} dx dy = \int_0^1 N(1, y) dy - \int_0^1 N(0, y) dy.$$

Let $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = (1, 1)$, and $P_4 = (0, 1)$. The first integral in the right-hand side equals the vector integral of the field \mathbf{F} over the segment P_2P_3 . The second integral equals the integral of \mathbf{F} over the segment P_1P_4 . Also, the integral of \mathbf{F} over any horizontal segment is 0. It follows that the entire right-hand side equals the integral of \mathbf{F} over the broken line $P_1P_2P_3P_4P_1$, that is, over ∂D .

Divergence Theorem

Theorem Let $D \subset \mathbb{R}^2$ be a closed, bounded region with piecewise smooth boundary ∂D oriented so that D is on the left as one traverses ∂D . Then for any smooth vector field \mathbf{F} on D ,

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \nabla \cdot \mathbf{F} \, dA.$$

Proof: Let \mathcal{L} denote the rotation of the plane \mathbb{R}^2 by 90° about the origin (counterclockwise). \mathcal{L} is a linear transformation preserving the dot product. Therefore

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \oint_{\partial D} \mathcal{L}(\mathbf{F}) \cdot \mathcal{L}(\mathbf{n}) \, ds.$$

Note that $\mathcal{L}(\mathbf{n})$ is the unit tangent vector to ∂D . It follows that the right-hand side is the vector integral of $\mathcal{L}(\mathbf{F})$ over ∂D . If $\mathbf{F} = (M, N)$ then $\mathcal{L}(\mathbf{F}) = (-N, M)$. By Green's Theorem,

$$\oint_{\partial D} \mathcal{L}(\mathbf{F}) \cdot d\mathbf{s} = \oint_{\partial D} -N \, dx + M \, dy = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy.$$