

MATH 311

Topics in Applied Mathematics I

Lecture 16:

Basis and dimension (continued).

Rank of a matrix.

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Theorem A nonempty set $S \subset V$ is a basis for V if and only if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination

$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \dots, r_k \in \mathbb{R}$.

Remark on uniqueness. Expansions $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{v} = -\mathbf{v}_2 + 2\mathbf{v}_1$, and $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2 + 0\mathbf{v}_3$ are considered the same.

Dimension

Theorem 1 Any vector space has a basis.

Theorem 2 If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V , denoted $\dim V$, is the number of elements in any of its bases.

Examples. • $\dim \mathbb{R}^n = n$

- $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices; $\dim \mathcal{M}_{m,n} = mn$
- \mathcal{P}_n : polynomials of degree less than n ; $\dim \mathcal{P}_n = n$
- \mathcal{P} : the space of all polynomials; $\dim \mathcal{P} = \infty$
- $\{\mathbf{0}\}$: the trivial vector space; $\dim \{\mathbf{0}\} = 0$

How to find a basis?

Theorem Let V be a vector space. Then

- (i) any spanning set for V contains a basis;
- (ii) any linearly independent subset of V is contained in a basis.

Approach 1. Given a spanning set for the vector space, reduce this set to a basis.

Approach 2. Given a linearly independent set, extend this set to a basis.

How to find a basis?

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis dropping one vector at a time.

Proposition Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ be a spanning set for a vector space V . If \mathbf{v}_0 is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for V .

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \dots + r_k\mathbf{v}_k$, then

$$\begin{aligned} t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k &= \\ &= (t_0r_1 + t_1)\mathbf{v}_1 + \dots + (t_0r_k + t_k)\mathbf{v}_k. \end{aligned}$$

How to find a basis?

Approach 2. Build a maximal linearly independent set adding one vector at a time.

If the vector space V is trivial, it has the empty basis. If $V \neq \{\mathbf{0}\}$, pick any vector $\mathbf{v}_1 \neq \mathbf{0}$. If \mathbf{v}_1 spans V , it is a basis. Otherwise pick any vector $\mathbf{v}_2 \in V$ that is not in the span of \mathbf{v}_1 . If \mathbf{v}_1 and \mathbf{v}_2 span V , they constitute a basis. Otherwise pick any vector $\mathbf{v}_3 \in V$ that is not in the span of \mathbf{v}_1 and \mathbf{v}_2 . And so on...

Modifications. Instead of the empty set, we can start with any linearly independent set (if we are given one). If we are given a spanning set S , it is enough to pick new vectors only in S .

Remark. This inductive procedure works for finite-dimensional vector spaces. There is an analogous procedure for infinite-dimensional spaces (*transfinite induction*).

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for \mathbb{R}^3 .

Our task is to find a vector \mathbf{v}_3 that is not a linear combination of \mathbf{v}_1 and \mathbf{v}_2 .

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for \mathbb{R}^3 .

Hint 1. \mathbf{v}_1 and \mathbf{v}_2 span the plane $x + 2z = 0$.

The vector $\mathbf{v}_3 = (1, 1, 1)$ does not lie in the plane $x + 2z = 0$, hence it is not a linear combination of \mathbf{v}_1 and \mathbf{v}_2 . Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 .

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Our task is to find a vector \mathbf{v}_3 that is not a linear combination of \mathbf{v}_1 and \mathbf{v}_2 . Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for \mathbb{R}^3 .

Hint 2. Since vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form a spanning set for \mathbb{R}^3 , at least one of them can be chosen as \mathbf{v}_3 .

Let us check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ are two bases for \mathbb{R}^3 :

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \quad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

To pare this spanning set, we need to find a relation of the form $r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + r_3\mathbf{w}_3 + r_4\mathbf{w}_4 = \mathbf{0}$, where $r_i \in \mathbb{R}$ are not all equal to zero. Equivalently,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system of linear equations for r_1, r_2, r_3, r_4 , we apply row reduction.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 \rightarrow \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{reduced row echelon form})$$

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \iff \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

General solution: $(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$, $t \in \mathbb{R}$.

Particular solution: $(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$.

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

We have obtained that $2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$.

Hence any of vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ can be dropped.

For instance, $V = \text{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4)$.

Let us check whether vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4$ are linearly independent:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0.$$

They are!!! It follows that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ is a basis for V . Also, it follows that $V = \mathbb{R}^3$.

Row space of a matrix

Definition. The **row space** of an $m \times n$ matrix A is the subspace of \mathbb{R}^n spanned by rows of A .

The dimension of the row space is called the **rank** of the matrix A .

Theorem 1 The rank of a matrix A is the maximal number of linearly independent rows in A .

Theorem 2 Elementary row operations do not change the row space of a matrix.

Theorem 3 If a matrix A is in row echelon form, then the nonzero rows of A are linearly independent.

Corollary The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

Problem. Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Elementary row operations do not change the row space. Let us convert A to row echelon form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for the row space of A . Thus the rank of A is 3.

It follows that the row space of A is the entire space \mathbb{R}^3 .

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

The vector space V is the row space of a matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

According to the solution of the previous problem, vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for V .

Column space of a matrix

Definition. The **column space** of an $m \times n$ matrix A is the subspace of \mathbb{R}^m spanned by columns of A .

Theorem 1 The column space of a matrix A coincides with the row space of the transpose matrix A^T .

Theorem 2 Elementary row operations do not change linear relations between columns of a matrix.

Theorem 3 Elementary row operations do not change the dimension of the column space of a matrix (however they can change the column space).

Theorem 4 If a matrix is in row echelon form, then the columns with leading entries form a basis for the column space.

Corollary For any matrix, the row space and the column space have the same dimension.

Problem. Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The column space of A coincides with the row space of A^T .
To find a basis, we convert A^T to row echelon form:

$$A^T = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Vectors $(1, 0, 2, 1)$, $(0, 1, 1, 0)$, and $(0, 0, 0, 1)$ form a basis for the column space of A .

Problem. Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Alternative solution: We already know from a previous problem that the rank of A is 3. It follows that the columns of A are linearly independent. Therefore these columns form a basis for the column space.

Problem. Let V be a vector space spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$. Pare this spanning set to a basis for V .

Alternative solution: The vector space V is the column space of a matrix

$$B = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The row echelon form of B is $C = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Columns of C with leading entries (1st, 2nd, and 4th) form a basis for the column space of C . It follows that the corresponding columns of B (i.e., 1st, 2nd, and 4th) form a basis for the column space of B .

Thus $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ is a basis for V .