# MATH 311

# Topics in Applied Mathematics I

## Lecture 24:
## Diagonalization.
## Euclidean structure in $\mathbb{R}^n$.

# Diagonalization

**Theorem 1** Let $L$ be a linear operator on a finite-dimensional vector space $V$. Then the following conditions are equivalent:

- the matrix of $L$ with respect to some basis is diagonal;
- there exists a basis for $V$ formed by eigenvectors of $L$.

The operator $L$ is **diagonalizable** if it satisfies these conditions.

**Theorem 2** Let $A$ be an $n \times n$ matrix. Then the following conditions are equivalent:

- $A$ is the matrix of a diagonalizable operator;
- $A$ is similar to a diagonal matrix, i.e., it is represented as $A = UBU^{-1}$, where the matrix $B$ is diagonal;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of $A$.

The matrix $A$ is **diagonalizable** if it satisfies these conditions.

To *diagonalize* an $n \times n$ matrix $A$ is to find a diagonal matrix $B$ and an invertible matrix $U$ such that $A = UBU^{-1}$.

Suppose there exists a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for $\mathbb{R}^n$ consisting of eigenvectors of $A$. That is, $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, where $\lambda_k \in \mathbb{R}$.

Then $A = UBU^{-1}$, where $B = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $U$ is a transition matrix whose columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

*Example.* $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. $\quad \det(A - \lambda I) = (4 - \lambda)(1 - \lambda)$.

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 1$.

Associated eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Thus $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Suppose we have a problem that involves a square matrix $A$ in the context of matrix multiplication.

Also, suppose that the case when $A$ is a diagonal matrix is simple. Then the diagonalization may help in solving this problem (or may not). Namely, it may reduce the case of a diagonalizable matrix to that of a diagonal one.

An example of such a problem is, given a square matrix $A$, to find its power $A^k$:

$$A = \begin{pmatrix} s_1 & & & O \\ & s_2 & & \\ & & \ddots & \\ O & & & s_n \end{pmatrix} \implies A^k = \begin{pmatrix} s_1^k & & & O \\ & s_2^k & & \\ & & \ddots & \\ O & & & s_n^k \end{pmatrix}$$

**Problem.** Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find $A^5$.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then $A^5 = UBU^{-1}UBU^{-1}UBU^{-1}UBU^{-1}UBU^{-1}$

$$= UB^5U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1024 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1024 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1024 & 1023 \\ 0 & 1 \end{pmatrix}.$$

**Problem.** Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find $A^k$ $(k \geq 1)$.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A^k = UB^kU^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4^k & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4^k & 4^k - 1 \\ 0 & 1 \end{pmatrix}.$$

**Problem.** Let $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Find a matrix $C$ such that $C^2 = A$.

We know that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Suppose that $D^2 = B$ for some matrix $D$. Let $C = UDU^{-1}$.
Then $C^2 = UDU^{-1}UDU^{-1} = UD^2U^{-1} = UBU^{-1} = A$.

We can take $D = \begin{pmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{1} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

Initial value problem for a system of linear ODEs:

$$\begin{cases} \frac{dx}{dt} = 4x + 3y, \\ \frac{dy}{dt} = y, \end{cases} \qquad x(0) = 1, \;\; y(0) = 1.$$

The system can be rewritten in vector form:

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v}, \quad \text{where } A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, \; \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Matrix $A$ is diagonalizable: $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be coordinates of the vector $\mathbf{v}$ relative to the basis $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (-1, 1)$ of eigenvectors of $A$. Then $\mathbf{v} = U\mathbf{w} \implies \mathbf{w} = U^{-1}\mathbf{v}$.

It follows that

$$\frac{d\mathbf{w}}{dt} = \frac{d}{dt}(U^{-1}\mathbf{v}) = U^{-1}\frac{d\mathbf{v}}{dt} = U^{-1}A\mathbf{v} = U^{-1}AU\mathbf{w}.$$

Hence $\quad \dfrac{d\mathbf{w}}{dt} = B\mathbf{w} \quad \Longleftrightarrow \quad \begin{cases} \frac{dw_1}{dt} = 4w_1, \\ \frac{dw_2}{dt} = w_2. \end{cases}$

General solution: $w_1(t) = c_1 e^{4t}$, $w_2(t) = c_2 e^t$, where $c_1, c_2 \in \mathbb{R}$.

Initial condition:

$$\mathbf{w}(0) = U^{-1}\mathbf{v}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus $w_1(t) = 2e^{4t}$, $w_2(t) = e^t$. Then

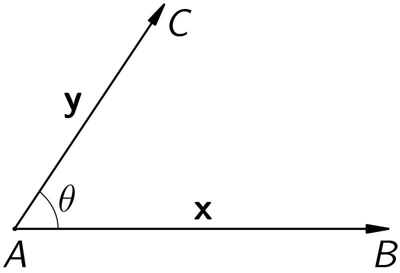
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = U\mathbf{w}(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2e^{4t} \\ e^t \end{pmatrix} = \begin{pmatrix} 2e^{4t} - e^t \\ e^t \end{pmatrix}.$$

## Beyond linearity: Euclidean structure

The vector space $\mathbb{R}^n$ is also a **Euclidean space**.

The Euclidean structure includes:

- length of a vector: $|\mathbf{x}|$,
- angle between vectors: $\theta$,
- dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\cos\theta$.



# Length and distance

*Definition.* The **length** of a vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The **distance** between vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as $\|\mathbf{y} - \mathbf{x}\|$.

*Properties of length:*

$\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$\|r\mathbf{x}\| = |r| \, \|\mathbf{x}\|$ (homogeneity)

$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

# Scalar product

*Definition.* The **scalar product** of vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is

$$\boxed{\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.}$$

Alternative notation: $(\mathbf{x}, \mathbf{y})$ or $\langle \mathbf{x}, \mathbf{y} \rangle$.

*Properties of scalar product:*

$\mathbf{x} \cdot \mathbf{x} \geq 0$, $\mathbf{x} \cdot \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (symmetry)

$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$ (distributive law)

$(r\mathbf{x}) \cdot \mathbf{y} = r(\mathbf{x} \cdot \mathbf{y})$ (homogeneity)

In particular, $\mathbf{x} \cdot \mathbf{y}$ is a **bilinear** function (i.e., it is both a linear function of $\mathbf{x}$ and a linear function of $\mathbf{y}$).

# Angle

*Cauchy-Schwarz inequality:* $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$.

By the Cauchy-Schwarz inequality, for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \quad \text{for a unique} \quad 0 \leq \theta \leq \pi.$$

$\theta$ is called the **angle** between the vectors $\mathbf{x}$ and $\mathbf{y}$.

The vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\mathbf{x} \cdot \mathbf{y} = 0$ (i.e., if $\theta = 90^{\circ}$).