MATH 311

Topics in Applied Mathematics I

Lecture 33:
Review of integral calculus (continued).
Area and volume.

Change of the variable in an integral

Theorem If ϕ is continuously differentiable on a closed interval $[a, b]$ and f is continuous on $\phi([a, b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)\, dt = \int_a^b f(\phi(x))\, \phi'(x)\, dx = \int_a^b f(\phi(x))\, d\phi(x).$$

Remarks. • The **Leibniz differential** $d\phi$ of the function ϕ is defined by $d\phi(x) = \phi'(x)\, dx = \frac{d\phi}{dx}\, dx$.

• It is possible that $\phi(a) \geq \phi(b)$. Hence we set

$$\int_c^d f(t)\, dt = -\int_d^c f(t)\, dt$$

if $c > d$. Also, we set the integral to be 0 if $c = d$.

• $t = \phi(x)$ is a proper change of the variable only if the function ϕ is strictly monotone. However the theorem holds even without this assumption.

Problem. Evaluate $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$.

To integrate this function, we introduce a new variable $u = 1 - x^2$:

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_0^{1/2} \frac{(1-x^2)'}{\sqrt{1-x^2}}\,dx$$

$$= -\frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,d(1-x^2) = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du$$

$$= \int_{3/4}^{1} \frac{1}{2\sqrt{u}}\,du = \sqrt{u}\,\Big|_{u=3/4}^{1} = 1 - \frac{\sqrt{3}}{2}.$$

Sets of measure zero

Definition. A subset E of the real line $\mathbb{R}$ is said to have **measure zero** if for any $\varepsilon > 0$ the set E can be covered by a sequence of open intervals $J_1, J_2, \dots$ such that $\sum_{n=1}^{\infty} |J_n| < \varepsilon$.

Examples. • Any set E that can be represented as a sequence $x_1, x_2, \dots$ (such sets are called **countable**) has measure zero. Indeed, for any $\varepsilon > 0$, let

$$J_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, \, x_n + \frac{\varepsilon}{2^{n+1}} \right), \quad n = 1, 2, \dots$$

Then $E \subset J_1 \cup J_2 \cup \dots$ and $|J_n| = \varepsilon / 2^n$ for all $n \in \mathbb{N}$ so that $\sum_{n=1}^{\infty} |J_n| = \varepsilon$.

• The set $\mathbb{Q}$ of rational numbers has measure zero (since it is countable).

• Nondegenerate interval $[a, b]$ is not a set of measure zero.

Lebesgue's criterion for Riemann integrability

Definition. Suppose $P(x)$ is a property depending on $x \in S$, where $S \subset \mathbb{R}$. We say that $P(x)$ holds for **almost all** $x \in S$ (or **almost everywhere** on S) if the set $\{x \in S \mid P(x) \text{ does not hold}\}$ has measure zero.

Theorem A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on the interval $[a, b]$ if and only if f is bounded on $[a, b]$ and continuous almost everywhere on $[a, b]$.

Let $\mathcal{P}$ be the smallest collection of subsets of $\mathbb{R}^2$ such that it contains all polygons and if $X, Y \in \mathcal{P}$, then $X \cup Y, X \cap Y, X \setminus Y \in \mathcal{P}$.

Theorem There exists a unique function $\mu : \mathcal{P} \to \mathbb{R}$ (called the **area function**) that satisfies the following conditions:

- **(positivity)** $\mu(X) \geq 0$ for all $X \in \mathcal{P}$;
- **(additivity)** $\mu(X \cup Y) = \mu(X) + \mu(Y)$ if $X \cap Y = \emptyset$;
- **(translation invariance)** $\mu(X + \mathbf{v}) = \mu(X)$ for all $X \in \mathcal{P}$ and $\mathbf{v} \in \mathbb{R}^2$;
- $\mu(Q) = 1$, where $Q = [0,1] \times [0,1]$ is the unit square.

The area function satisfies an extra condition:

- **(monotonicity)** $\mu(X) \leq \mu(Y)$ whenever $X \subset Y$.

Now for any bounded set $X \subset \mathbb{R}^2$ we let $\overline{\mu}(X) = \inf_{X \subset Y} \mu(Y)$ and $\underline{\mu}(X) = \sup_{Z \subset X} \mu(Z)$. Note that $\underline{\mu}(X) \leq \overline{\mu}(X)$. In the case of equality, the set X is called **Jordan measurable** and we let $\mathrm{area}(X) = \overline{\mu}(X)$.

Area, volume, and determinants

- 2×2 determinants and plane geometry

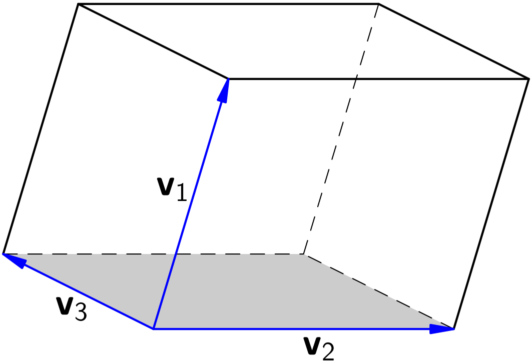
Let P be a parallelogram in the plane $\mathbb{R}^2$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are represented by adjacent sides of P. Then $\mathrm{area}(P) = |\det A|$, where $A = (\mathbf{v}_1, \mathbf{v}_2)$, a matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$.

Consider a linear operator $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L_A(\mathbf{v}) = A\mathbf{v}$ for any column vector $\mathbf{v}$. Then $\mathrm{area}(L_A(D)) = |\det A| \,\mathrm{area}(D)$ for any bounded domain D.

- 3×3 determinants and space geometry

Let Π be a parallelepiped in space $\mathbb{R}^3$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are represented by adjacent edges of Π. Then $\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, a matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Similarly, $\mathrm{volume}(L_B(D)) = |\det B| \,\mathrm{volume}(D)$ for any bounded domain $D \subset \mathbb{R}^3$.



$\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Note that the parallelepiped Π is the image under L_B of a unit cube whose adjacent edges are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the right-hand rule. We say that L_B **preserves orientation** if it preserves the hand rule for any basis. This is the case if and only if $\det B > 0$.

Riemann sums in two dimensions

Consider a closed coordinate rectangle
$R = [a, b] \times [c, d] \subset \mathbb{R}^2$.

Definition. A **Riemann sum** of a function $f : R \to \mathbb{R}$ with respect to a partition $P = \{D_1, D_2, \ldots, D_n\}$ of R generated by samples $t_j \in D_j$ is a sum

$$\mathcal{S}(f, P, t_j) = \sum\nolimits_{j=1}^{n} f(t_j)\ \mathrm{area}(D_j).$$

The norm of the partition P is $\|P\| = \max_{1 \le j \le n} \mathrm{diam}(D_j)$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

If this is the case, then the function f is called **integrable** on R and the limit $I(f)$ is called the **integral** of f over R.

Double integral

Closed coordinate rectangle $R = [a, b] \times [c, d]$
$= \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \;\; c \leq y \leq d\}$.

Notation: $\displaystyle\iint_R f \, dA$ or $\displaystyle\iint_R f(x, y) \, dx \, dy$.

Theorem 1 If f is continuous on the closed rectangle R, then f is integrable.

Theorem 2 A function $f : R \to \mathbb{R}$ is Riemann integrable on the rectangle R if and only if f is bounded on R and continuous almost everywhere on R (that is, the set of discontinuities of f has zero area).

Fubini's Theorem

Fubini's Theorem allows us to reduce a multiple integral to a repeated one-dimensional integral.

Theorem If a function f is integrable on $R = [a, b] \times [c, d]$, then

$$\iint_R f\,dA = \int_a^b \Big(\int_c^d f(x, y)\,dy\Big)\,dx = \int_c^d \Big(\int_a^b f(x, y)\,dx\Big)\,dy.$$

In particular, this implies that we can change the order of integration in a repeated integral.

Corollary If a function g is integrable on $[a, b]$ and a function h is integrable on $[c, d]$, then the function $f(x, y) = g(x)h(y)$ is integrable on $R = [a, b] \times [c, d]$ and

$$\iint_R g(x)h(y)\,dx\,dy = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy.$$

Integrals over general domains

Suppose $f : D \to \mathbb{R}$ is a function defined on a (Jordan) measurable set $D \subset \mathbb{R}^2$. Since D is bounded, it is contained in a rectangle R. To define the integral of f over D, we extend the function f to a function on R:

$$f^{\text{ext}}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{if } (x, y) \notin D. \end{cases}$$

Definition. $\iint_D f \, dA$ is defined to be $\iint_R f^{\text{ext}} \, dA$.

In particular, $\text{area}(D) = \iint_D 1 \, dA$.