

MATH 311

Topics in Applied Mathematics I

Lecture 13:

Linear transformations (continued).

General linear equations.

Matrix representation of linear maps.

Linear transformation

Definition. Given vector spaces V_1 and V_2 , a mapping $L : V_1 \rightarrow V_2$ is **linear** if

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = rL(\mathbf{x})$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear mappings:

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$
for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.
- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2 , respectively.
- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

Examples of linear mappings

- *Scaling* $L : V \rightarrow V$, $L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{R}$.

$$L(\mathbf{x} + \mathbf{y}) = s(\mathbf{x} + \mathbf{y}) = s\mathbf{x} + s\mathbf{y} = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = s(r\mathbf{x}) = (sr)\mathbf{x} = (rs)\mathbf{x} = r(s\mathbf{x}) = rL(\mathbf{x}).$$

- *Dot product with a fixed vector*

$$\ell : \mathbb{R}^n \rightarrow \mathbb{R}, \ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0, \text{ where } \mathbf{v}_0 \in \mathbb{R}^n.$$

$$\ell(\mathbf{x} + \mathbf{y}) = (\mathbf{x} + \mathbf{y}) \cdot \mathbf{v}_0 = \mathbf{x} \cdot \mathbf{v}_0 + \mathbf{y} \cdot \mathbf{v}_0 = \ell(\mathbf{x}) + \ell(\mathbf{y}),$$

$$\ell(r\mathbf{x}) = (r\mathbf{x}) \cdot \mathbf{v}_0 = r(\mathbf{x} \cdot \mathbf{v}_0) = r\ell(\mathbf{x}).$$

- *Cross product with a fixed vector*

$$L : \mathbb{R}^3 \rightarrow \mathbb{R}^3, L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0, \text{ where } \mathbf{v}_0 \in \mathbb{R}^3.$$

- *Multiplication by a fixed matrix*

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^m, L(\mathbf{v}) = A\mathbf{v}, \text{ where } A \text{ is an } m \times n \text{ matrix and all vectors are column vectors.}$$

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$$\ell : F(\mathbb{R}) \rightarrow \mathbb{R}, \quad \ell(f) = f(a), \quad \text{where } a \in \mathbb{R}.$$

- *Multiplication by a fixed function*

$$L : F(\mathbb{R}) \rightarrow F(\mathbb{R}), \quad L(f) = gf, \quad \text{where } g \in F(\mathbb{R}).$$

- *Differentiation* $D : C^1(\mathbb{R}) \rightarrow C(\mathbb{R}), \quad D(f) = f'.$

$$D(f + g) = (f + g)' = f' + g' = D(f) + D(g),$$

$$D(rf) = (rf)' = rf' = rD(f).$$

- *Integration over a finite interval*

$$\ell : C(\mathbb{R}) \rightarrow \mathbb{R}, \quad \ell(f) = \int_a^b f(x) dx, \quad \text{where}$$

$$a, b \in \mathbb{R}, \quad a < b.$$

More properties of linear mappings

- If a linear mapping $L : V \rightarrow W$ is invertible then the inverse mapping $L^{-1} : W \rightarrow V$ is also linear.
- If $L : V \rightarrow W$ and $M : W \rightarrow X$ are linear mappings then the composition $M \circ L : V \rightarrow X$ is also linear.
- If $L_1 : V \rightarrow W$ and $L_2 : V \rightarrow W$ are linear mappings then the sum $L_1 + L_2$ is also linear.

Linear differential operators

- *Ordinary differential operator*

$$L : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on \mathbb{R} .

That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- *Laplace's operator* $\Delta : C^\infty(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Linear integral operators

- *Anti-derivative*

$$L : C[a, b] \rightarrow C^1[a, b], \quad (Lf)(x) = \int_a^x f(y) dy.$$

- *Hilbert-Schmidt operator*

$$L : C[a, b] \rightarrow C[c, d], \quad (Lf)(x) = \int_a^b K(x, y)f(y) dy,$$

where $K \in C([c, d] \times [a, b])$.

- *Laplace transform*

$$\mathcal{L} : BC(0, \infty) \rightarrow C(0, \infty), \quad (\mathcal{L}f)(x) = \int_0^{\infty} e^{-xy} f(y) dy.$$

Examples. $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices.

- $\alpha : \mathcal{M}_{m,n}(\mathbb{R}) \rightarrow \mathcal{M}_{n,m}(\mathbb{R}), \alpha(A) = A^T.$

$$\alpha(A + B) = \alpha(A) + \alpha(B) \iff (A + B)^T = A^T + B^T.$$

$$\alpha(rA) = r\alpha(A) \iff (rA)^T = rA^T.$$

Hence α is linear.

- $\beta : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathbb{R}, \beta(A) = \det A.$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$

Then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

We have $\det(A) = \det(B) = 0$ while $\det(A + B) = 1.$

Hence $\beta(A + B) \neq \beta(A) + \beta(B)$ so that β is not linear.

Range and kernel

Let V, W be vector spaces and $L : V \rightarrow W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L , denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem (i) The range of L is a subspace of W .
(ii) The kernel of L is a subspace of V .

Example. $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The kernel $\ker(L)$ is the nullspace of the matrix.

$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The range $L(\mathbb{R}^3)$ is the column space of the matrix.

Example. $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The range of L is spanned by vectors $(1, 1, 1)$, $(0, 2, 0)$, and $(-1, -1, -1)$. It follows that $L(\mathbb{R}^3)$ is the plane spanned by $(1, 1, 1)$ and $(0, 1, 0)$.

To find $\ker(L)$, we apply row reduction to the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(x, y, z) \in \ker(L)$ if $x - z = y = 0$.

It follows that $\ker(L)$ is the line spanned by $(1, 0, 1)$.

Example. $L: C^3(\mathbb{R}) \rightarrow C(\mathbb{R}), L(u) = u''' - 2u'' + u'$.

According to the theory of differential equations, the initial value problem

$$\begin{cases} u'''(x) - 2u''(x) + u'(x) = g(x), & x \in \mathbb{R}, \\ u(a) = b_0, \\ u'(a) = b_1, \\ u''(a) = b_2 \end{cases}$$

has a unique solution for any $g \in C(\mathbb{R})$ and any $b_0, b_1, b_2 \in \mathbb{R}$. It follows that $L(C^3(\mathbb{R})) = C(\mathbb{R})$.

Also, the initial data evaluation $I(u) = (u(a), u'(a), u''(a))$, which is a linear mapping $I: C^3(\mathbb{R}) \rightarrow \mathbb{R}^3$, becomes invertible when restricted to $\ker(L)$. Hence $\dim \ker(L) = 3$.

It is easy to check that $L(xe^x) = L(e^x) = L(1) = 0$.

Besides, the functions xe^x , e^x , and 1 are linearly independent (use Wronskian). It follows that $\ker(L) = \text{Span}(xe^x, e^x, 1)$.

General linear equation

Definition. A **linear equation** is an equation of the form

$$L(\mathbf{x}) = \mathbf{b},$$

where $L : V \rightarrow W$ is a linear mapping, \mathbf{b} is a given vector from W , and \mathbf{x} is an unknown vector from V .

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The kernel of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable and $\dim \ker L < \infty$, then the general solution is

$$\mathbf{x}_0 + t_1 \mathbf{v}_1 + \cdots + t_k \mathbf{v}_k,$$

where \mathbf{x}_0 is a particular solution, $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a basis for the kernel of L , and t_1, \dots, t_k are arbitrary scalars.

Example.
$$\begin{cases} x + y + z = 4, \\ x + 2y = 3. \end{cases}$$

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Linear equation: $L(\mathbf{x}) = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 0 & 3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -1 \end{array} \right)$$

$$\begin{cases} x + 2z = 5 \\ y - z = -1 \end{cases} \iff \begin{cases} x = 5 - 2z \\ y = -1 + z \end{cases}$$

$$(x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1).$$

Example. $u'''(x) - 2u''(x) + u'(x) = e^{2x}$.

Linear operator $L : C^3(\mathbb{R}) \rightarrow C(\mathbb{R})$,

$$Lu = u''' - 2u'' + u'.$$

Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

We already know that functions xe^x , e^x and 1 form a basis for the kernel of L . It remains to find a particular solution.

$$L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x}.$$

Since L is a linear operator, $L\left(\frac{1}{2}e^{2x}\right) = e^{2x}$.

Particular solution: $u_0(x) = \frac{1}{2}e^{2x}$.

Thus the general solution is

$$u(x) = \frac{1}{2}e^{2x} + t_1xe^x + t_2e^x + t_3.$$

Matrix transformations

Any $m \times n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $L(\mathbf{x}) \in \mathbb{R}^m$ are regarded as column vectors. This transformation is **linear**.

Example.
$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the standard basis for \mathbb{R}^3 . We have that $L(\mathbf{e}_1) = (1, 3, 0)$, $L(\mathbf{e}_2) = (0, 4, 5)$, $L(\mathbf{e}_3) = (2, 7, 8)$. Thus $L(\mathbf{e}_1)$, $L(\mathbf{e}_2)$, $L(\mathbf{e}_3)$ are columns of the matrix.

Problem. Find a linear mapping $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $L(\mathbf{e}_1) = (1, 1)$, $L(\mathbf{e}_2) = (0, -2)$, $L(\mathbf{e}_3) = (3, 0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for \mathbb{R}^3 .

$$\begin{aligned}L(x, y, z) &= L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) \\ &= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3) \\ &= x(1, 1) + y(0, -2) + z(3, 0) = (x + 3z, x - 2y)\end{aligned}$$

$$L(x, y, z) = \begin{pmatrix} x + 3z \\ x - 2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem Suppose $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for \mathbb{R}^n .

$$\mathbf{y} = A\mathbf{x} \iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

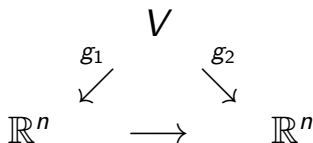
$$\iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Change of coordinates (revisited)

Let V be a vector space.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.



The composition $g_2 \circ g_1^{-1}$ is a linear mapping of \mathbb{R}^n to itself. Hence it's represented as $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

Matrix of a linear transformation

Let V, W be vector spaces and $f : V \rightarrow W$ be a linear map.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ be a basis for W and $g_2 : W \rightarrow \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ g_1 \downarrow & & \downarrow g_2 \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of \mathbb{R}^n to \mathbb{R}^m . Hence it's represented as $\mathbf{x} \mapsto A\mathbf{x}$, where A is an $m \times n$ matrix.

A is called the **matrix of f** with respect to bases $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \dots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \dots, \mathbf{w}_m$.

Examples. • $D : \mathcal{P}_3 \rightarrow \mathcal{P}_2$, $(Dp)(x) = p'(x)$.

Let A_D be the matrix of D with respect to the bases $1, x, x^2$ and $1, x$. Columns of A_D are coordinates of polynomials $D1, Dx, Dx^2$ w.r.t. the basis $1, x$.

$$D1 = 0, Dx = 1, Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

• $L : \mathcal{P}_3 \rightarrow \mathcal{P}_3$, $(Lp)(x) = p(x + 1)$.

Let A_L be the matrix of L w.r.t. the basis $1, x, x^2$.

$L1 = 1$, $Lx = 1 + x$, $Lx^2 = (x + 1)^2 = 1 + 2x + x^2$.

$$\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$