

MATH 311

Topics in Applied Mathematics I

Lecture 15:
Eigenvalues and eigenvectors.
Characteristic equation.

Eigenvalues and eigenvectors of a matrix

Definition. Let A be an $n \times n$ matrix. A number $\lambda \in \mathbb{R}$ is called an **eigenvalue** of the matrix A if $A\mathbf{v} = \lambda\mathbf{v}$ for a nonzero column vector $\mathbf{v} \in \mathbb{R}^n$.

The vector \mathbf{v} is called an **eigenvector** of A belonging to (or associated with) the eigenvalue λ .

Remarks.

- Alternative notation:
eigenvalue = **characteristic value**,
eigenvector = **characteristic vector**.

- The zero vector is never considered an eigenvector.

Example. $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Hence $(1, 0)$ is an eigenvector of A belonging to the eigenvalue 2, while $(0, -2)$ is an eigenvector of A belonging to the eigenvalue 3.

Example. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence $(1, 1)$ is an eigenvector of A belonging to the eigenvalue 1, while $(1, -1)$ is an eigenvector of A belonging to the eigenvalue -1 .

Vectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, -1)$ form a basis for \mathbb{R}^2 . Consider a linear operator $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $L(\mathbf{x}) = A\mathbf{x}$. The matrix of L with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$ is $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let A be an $n \times n$ matrix. Consider a linear operator $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $L(\mathbf{x}) = A\mathbf{x}$.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a nonstandard basis for \mathbb{R}^n and B be the matrix of the operator L with respect to this basis.

Theorem The matrix B is diagonal if and only if vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are eigenvectors of A .

If this is the case, then the diagonal entries of the matrix B are the corresponding eigenvalues of A .

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i \iff B = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Eigenspaces

Let A be an $n \times n$ matrix. Let \mathbf{v} be an eigenvector of A belonging to an eigenvalue λ .

Then $A\mathbf{v} = \lambda\mathbf{v} \implies A\mathbf{v} = (\lambda I)\mathbf{v} \implies (A - \lambda I)\mathbf{v} = \mathbf{0}$.

Hence $\mathbf{v} \in N(A - \lambda I)$, the nullspace of the matrix $A - \lambda I$.

Conversely, if $\mathbf{x} \in N(A - \lambda I)$ then $A\mathbf{x} = \lambda\mathbf{x}$.

Thus the eigenvectors of A belonging to the eigenvalue λ are nonzero vectors from $N(A - \lambda I)$.

Definition. If $N(A - \lambda I) \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of the matrix A corresponding to the eigenvalue λ .

How to find eigenvalues and eigenvectors?

Theorem Given a square matrix A and a scalar λ , the following statements are equivalent:

- λ is an eigenvalue of A ,
- $N(A - \lambda I) \neq \{\mathbf{0}\}$,
- the matrix $A - \lambda I$ is singular,
- $\det(A - \lambda I) = 0$.

Definition. $\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix A .

Eigenvalues λ of A are roots of the characteristic equation. Associated eigenvectors of A are nonzero solutions of the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Example. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc). \end{aligned}$$

Example. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$
$$= -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3,$$

where $c_1 = a_{11} + a_{22} + a_{33}$ (the *trace* of A),

$$c_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$$c_3 = \det A.$$

Theorem. Let $A = (a_{ij})$ be an $n \times n$ matrix.

Then $\det(A - \lambda I)$ is a polynomial of λ of degree n :

$$\det(A - \lambda I) = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n.$$

Furthermore, $(-1)^{n-1} c_1 = a_{11} + a_{22} + \cdots + a_{nn}$
and $c_n = \det A$.

Definition. The polynomial $p(\lambda) = \det(A - \lambda I)$ is called the **characteristic polynomial** of the matrix A .

Corollary Any $n \times n$ matrix has at most n eigenvalues.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Characteristic equation: $\begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = 0$.

$$(2 - \lambda)^2 - 1 = 0 \implies \lambda_1 = 1, \lambda_2 = 3.$$

$$(A - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + y = 0.$$

The general solution is $(-t, t) = t(-1, 1)$, $t \in \mathbb{R}$.

Thus $\mathbf{v}_1 = (-1, 1)$ is an eigenvector associated with the eigenvalue 1. The corresponding eigenspace is the line spanned by \mathbf{v}_1 .

$$(A - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\iff \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x - y = 0.$$

The general solution is $(t, t) = t(1, 1)$, $t \in \mathbb{R}$.

Thus $\mathbf{v}_2 = (1, 1)$ is an eigenvector associated with the eigenvalue 3. The corresponding eigenspace is the line spanned by \mathbf{v}_2 .

Summary. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 1 and 3.
- The eigenspace of A associated with the eigenvalue 1 is the line $t(-1, 1)$.
- The eigenspace of A associated with the eigenvalue 3 is the line $t(1, 1)$.
- Eigenvectors $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (1, 1)$ of the matrix A form a basis for \mathbb{R}^2 .
- Geometrically, the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is a stretch by a factor of 3 away from the line $x + y = 0$ in the orthogonal direction.

Example. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$

Characteristic equation:

$$\begin{vmatrix} 1 - \lambda & 1 & -1 \\ 1 & 1 - \lambda & 1 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = 0.$$

Expand the determinant by the 3rd row:

$$(2 - \lambda) \begin{vmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = 0.$$

$$((1 - \lambda)^2 - 1)(2 - \lambda) = 0 \iff -\lambda(2 - \lambda)^2 = 0$$

$$\implies \lambda_1 = 0, \lambda_2 = 2.$$

$$A\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Convert the matrix to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{0} \iff \begin{cases} x + y = 0, \\ z = 0. \end{cases}$$

The general solution is $(-t, t, 0) = t(-1, 1, 0)$, $t \in \mathbb{R}$. Thus $\mathbf{v}_1 = (-1, 1, 0)$ is an eigenvector associated with the eigenvalue 0. The corresponding eigenspace is the line spanned by \mathbf{v}_1 .

$$(A - 2I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x - y + z = 0.$$

The general solution is $x = t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. Equivalently,

$$\mathbf{x} = (t - s, t, s) = t(1, 1, 0) + s(-1, 0, 1).$$

Thus $\mathbf{v}_2 = (1, 1, 0)$ and $\mathbf{v}_3 = (-1, 0, 1)$ are eigenvectors associated with the eigenvalue 2.

The corresponding eigenspace is the plane spanned by \mathbf{v}_2 and \mathbf{v}_3 .

Summary. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 0 and 2.
- The eigenvalue 0 is *simple*: the corresponding eigenspace is a line.
- The eigenvalue 2 is of *multiplicity* 2: the corresponding eigenspace is a plane.
- Eigenvectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (-1, 0, 1)$ of the matrix A form a basis for \mathbb{R}^3 .
- Geometrically, the map $\mathbf{x} \mapsto A\mathbf{x}$ is the projection on the plane $\text{Span}(\mathbf{v}_2, \mathbf{v}_3)$ along the lines parallel to \mathbf{v}_1 with the subsequent scaling by a factor of 2.

Eigenvalues and eigenvectors of an operator

Definition. Let V be a vector space and $L : V \rightarrow V$ be a linear operator. A number λ is called an **eigenvalue** of the operator L if $L(\mathbf{v}) = \lambda\mathbf{v}$ for a nonzero vector $\mathbf{v} \in V$. The vector \mathbf{v} is called an **eigenvector** of L associated with the eigenvalue λ . (If V is a functional space then eigenvectors are also called **eigenfunctions**.)

If $V = \mathbb{R}^n$ then the linear operator L is given by $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix.

In this case, eigenvalues and eigenvectors of the operator L are precisely eigenvalues and eigenvectors of the matrix A .

Eigenspaces

Let $L : V \rightarrow V$ be a linear operator.

For any $\lambda \in \mathbb{R}$, let V_λ denotes the set of all solutions of the equation $L(\mathbf{x}) = \lambda\mathbf{x}$.

Then V_λ is a *subspace* of V since V_λ is the *kernel* of a linear operator given by $\mathbf{x} \mapsto L(\mathbf{x}) - \lambda\mathbf{x}$.

V_λ minus the zero vector is the set of all eigenvectors of L associated with the eigenvalue λ . In particular, $\lambda \in \mathbb{R}$ is an eigenvalue of L if and only if $V_\lambda \neq \{\mathbf{0}\}$.

If $V_\lambda \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of L corresponding to the eigenvalue λ .

Example. $V = C^\infty(\mathbb{R})$, $D : V \rightarrow V$, $Df = f'$.

A function $f \in C^\infty(\mathbb{R})$ is an eigenfunction of the operator D belonging to an eigenvalue λ if $f'(x) = \lambda f(x)$ for all $x \in \mathbb{R}$.

It follows that $f(x) = ce^{\lambda x}$, where c is a nonzero constant.

Thus each $\lambda \in \mathbb{R}$ is an eigenvalue of D .

The corresponding eigenspace is spanned by $e^{\lambda x}$.

Example. $V = C^\infty(\mathbb{R})$, $L : V \rightarrow V$, $Lf = f''$.

$$Lf = \lambda f \iff f''(x) - \lambda f(x) = 0 \text{ for all } x \in \mathbb{R}.$$

It follows that each $\lambda \in \mathbb{R}$ is an eigenvalue of L and the corresponding eigenspace V_λ is two-dimensional. Note that $L = D^2$, hence $Df = \mu f \implies Lf = \mu^2 f$.

If $\lambda > 0$ then $V_\lambda = \text{Span}(e^{\mu x}, e^{-\mu x})$, where $\mu = \sqrt{\lambda}$.

If $\lambda < 0$ then $V_\lambda = \text{Span}(\sin(\mu x), \cos(\mu x))$, where $\mu = \sqrt{-\lambda}$.

If $\lambda = 0$ then $V_\lambda = \text{Span}(1, x)$.