MATH 311

Topics in Applied Mathematics I

Lecture 20:
Review for Test 2.

Topics for Test 2

Vector spaces (Leon/Colley 3.5)

- Coordinates relative to a basis
- Change of basis, transition matrix

Linear transformations (Leon/Colley 4.1–4.3)

- Linear transformations
- Range and kernel
- Matrix of a linear transformation
- Change of basis for a linear operator
- Similar matrices

Topics for Test 2

Eigenvalues and eigenvectors (Leon/Colley 6.1, 6.3)

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Diagonalization

Orthogonality (Leon/Colley 5.1–5.6)

- Euclidean structure in $\mathbb{R}^n$
- Orthogonal complement
- Orthogonal projection
- Least squares problems
- Norms and inner products
- The Gram-Schmidt process

Sample problems for Test 2

Problem 1 Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2, \quad \text{where} \;\; \mathbf{v}_1 = (1,1,1), \;\; \mathbf{v}_2 = (1,2,2).$$

(i) Find the matrix of the operator L.
(ii) Find the dimensions of the range and the kernel of L.
(iii) Find bases for the range and the kernel of L.

Problem 2 Let V be a subspace of $F(\mathbb{R})$ spanned by functions e^x and e^{-x}. Let L be a linear operator on V such that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the matrix of L relative to the basis e^x, e^{-x}. Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Sample problems for Test 2

Problem 3 Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.
(ii) For each eigenvalue of A, find an associated eigenvector.
(iii) Is the matrix A diagonalizable? Explain.
(iv) Find all eigenvalues of the matrix A^2.

Problem 4 Find a linear polynomial which is the best least squares fit to the following data:

x	-2	-1	0	1	2
$f(x)$	-3	-2	1	2	5

Problem 5 Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$. Find the distance from the vector $\mathbf{y} = (1,0,0,0)$ to the subspaces V and $V^\perp$.

Problem 1. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2$, where $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 2, 2)$.

(i) Find the matrix of the operator L.

Let A denote the matrix of the linear operator L. The consecutive columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is the standard basis for $\mathbb{R}^3$.

Given $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{v} \cdot \mathbf{v}_1 = x + y + z$ and $L(\mathbf{v}) = \big(x + y + z, 2(x + y + z), 2(x + y + z)\big)$. It follows that $L(\mathbf{e}_1) = L(\mathbf{e}_2) = L(\mathbf{e}_3) = (1, 2, 2)$. Consequently,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Problem 1. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2$, where $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 2, 2)$.

(i) Find the matrix of the operator L.

Alternative solution: Given a vector $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, let $\alpha = \mathbf{v} \cdot \mathbf{v}_1$ and $(x_1, y_1, z_1) = L(\mathbf{v})$. In terms of matrix algebra, we have

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} (\alpha) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(note that scalar multiplication of a column vector is equivalent to multiplication by a 1×1 matrix but the matrix has to be on the right as otherwise the matrix product is not defined). It follows that the matrix of the operator L is

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Problem 1. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2$, where $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 2, 2)$.

(ii) Find the dimensions of the range and the kernel of L.
(iii) Find bases for the range and the kernel of L.

The range $\mathrm{Range}(L)$ of the linear operator L is the subspace of all vectors of the form $L(\mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^3$. It is easy to see that $\mathrm{Range}(L)$ is the line spanned by the vector $\mathbf{v}_2$. Hence $\dim \mathrm{Range}(L) = 1$ and $\mathbf{v}_2 = (1, 2, 2)$ forms a basis.

The kernel $\ker(L)$ of the operator L is the subspace of all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $L(\mathbf{x}) = \mathbf{0}$. Clearly, $L(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} \cdot \mathbf{v}_1 = 0$. Therefore $\ker(L)$ is the orthogonal complement of $\mathbf{v}_1$, the plane $x + y + z = 0$. The general solution of the equation is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. It gives rise to a parametric representation $t(-1, 1, 0) + s(-1, 0, 1)$ of the plane. Thus the kernel of L is spanned by vectors $(-1, 1, 0)$ and $(-1, 0, 1)$. Since the two vectors are linearly independent, they form a basis for $\ker(L)$ so that $\dim \ker(L) = 2$.

Problem 2. Let V be a subspace of $F(\mathbb{R})$ spanned by functions e^x and e^{-x}. Let L be a linear operator on V such that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the matrix of L relative to the basis e^x, e^{-x}. Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Let A denote the matrix of the operator L relative to the basis e^x, e^{-x} (which is given) and B denote the matrix of L relative to the basis $\cosh x$, $\sinh x$ (which is to be found). By definition of the functions $\cosh x$ and $\sinh x$, the transition matrix from $\cosh x, \sinh x$ to e^x, e^{-x} is $U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

It follows that $B = U^{-1}AU$. We obtain that

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}.$$

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.

The eigenvalues of A are roots of the characteristic equation $\det(A - \lambda I) = 0$. We obtain that

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^3 - 2(1-\lambda) - 2(1-\lambda) = (1-\lambda)\big((1-\lambda)^2 - 4\big)$$

$$= (1-\lambda)\big((1-\lambda) - 2\big)\big((1-\lambda) + 2\big) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix A has three eigenvalues: -1, 1, and 3.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(ii) For each eigenvalue of A, find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)$ of the matrix A associated with an eigenvalue λ is a nonzero solution of the vector equation

$$(A-\lambda I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve the equation, we convert the matrix $A - \lambda I$ to reduced row echelon form.

First consider the case $\lambda = -1$. The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A + I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of A associated with the eigenvalue -1.

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)$ is an eigenvector of A associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$A - 3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 0 & -1 \\ 0 & \color{red}{1} & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - 3I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of A associated with the eigenvalue 3.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iii) Is the matrix A diagonalizable? Explain.

The matrix A is diagonalizable, i.e., there exists a basis for $\mathbb{R}^3$ formed by its eigenvectors.

Namely, the vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix A belonging to distinct eigenvalues. Therefore these vectors are linearly independent. It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

Alternatively, the existence of a basis for $\mathbb{R}^3$ consisting of eigenvectors of A already follows from the fact that the matrix A has three distinct eigenvalues.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iv) Find all eigenvalues of the matrix A^2.

Suppose that $\mathbf{v}$ is an eigenvector of the matrix A associated with an eigenvalue λ, that is, $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

Therefore $\mathbf{v}$ is also an eigenvector of the matrix A^2 and the associated eigenvalue is λ^2. We already know that the matrix A has eigenvalues -1, 1, and 3. It follows that A^2 has eigenvalues 1 and 9.

Since a 3×3 matrix can have up to 3 eigenvalues, we need an additional argument to show that 1 and 9 are the only eigenvalues of A^2. One reason is that the eigenvalue 1 has multiplicity 2.

Problem 4. Find a linear polynomial which is the best least squares fit to the following data:

x	-2	-1	0	1	2
$f(x)$	-3	-2	1	2	5

We are looking for a function $f(x) = c_1 + c_2x$, where c_1, c_2 are unknown coefficients. The data of the problem give rise to an overdetermined system of linear equations in variables c_1 and c_2:

$$\begin{cases} c_1 - 2c_2 = -3, \\ c_1 - c_2 = -2, \\ c_1 = 1, \\ c_1 + c_2 = 2, \\ c_1 + 2c_2 = 5. \end{cases}$$

This system is inconsistent.

We can represent the system as a matrix equation $A\mathbf{c} = \mathbf{y}$, where

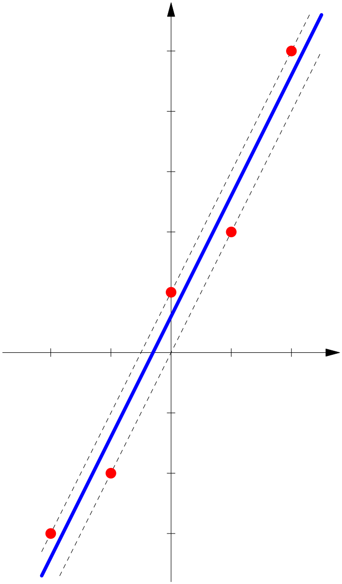
$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

The least squares solution $\mathbf{c}$ of the above system is a solution of the normal system $A^T A\mathbf{c} = A^T\mathbf{y}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\iff \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 20 \end{pmatrix} \iff \begin{cases} c_1 = 3/5 \\ c_2 = 2 \end{cases}$$

Thus the function $f(x) = \frac{3}{5} + 2x$ is the best least squares fit to the above data among linear polynomials.



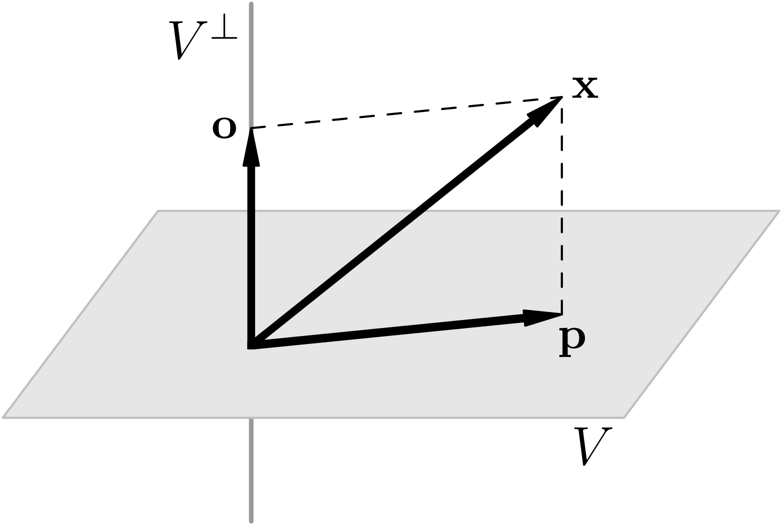
Problem 5. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$. Find the distance from the vector $\mathbf{y} = (1,0,0,0)$ to the subspaces V and $V^\perp$.

The vector $\mathbf{y}$ is uniquely decomposed as $\mathbf{y} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o} \in V^\perp$. Then $\mathbf{p}$ is the orthogonal projection of $\mathbf{y}$ onto the subspace V while $\mathbf{o}$ is the orthogonal projection of $\mathbf{y}$ onto the orthogonal complement $V^\perp$. Then the distance from $\mathbf{y}$ to V equals $\|\mathbf{y} - \mathbf{p}\| = \|\mathbf{o}\|$ and the distance from $\mathbf{y}$ to $V^\perp$ equals $\|\mathbf{y} - \mathbf{o}\| = \|\mathbf{p}\|$.

We have $\mathbf{p} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ for some $\alpha, \beta \in \mathbb{R}$. Then $\mathbf{o} = \mathbf{y} - \mathbf{p} = \mathbf{y} - \alpha\mathbf{x}_1 - \beta\mathbf{x}_2$. Since $\mathbf{o} \perp V$,

$$\begin{cases} \mathbf{o} \cdot \mathbf{x}_1 = 0 \\ \mathbf{o} \cdot \mathbf{x}_2 = 0 \end{cases} \iff \begin{cases} (\mathbf{y} - \alpha\mathbf{x}_1 - \beta\mathbf{x}_2) \cdot \mathbf{x}_1 = 0 \\ (\mathbf{y} - \alpha\mathbf{x}_1 - \beta\mathbf{x}_2) \cdot \mathbf{x}_2 = 0 \end{cases}$$

$$\iff \begin{cases} \alpha(\mathbf{x}_1 \cdot \mathbf{x}_1) + \beta(\mathbf{x}_2 \cdot \mathbf{x}_1) = \mathbf{y} \cdot \mathbf{x}_1 \\ \alpha(\mathbf{x}_1 \cdot \mathbf{x}_2) + \beta(\mathbf{x}_2 \cdot \mathbf{x}_2) = \mathbf{y} \cdot \mathbf{x}_2 \end{cases}$$



$V^\perp$
x
o
p
V

$\mathbf{y} = (1,0,0,0),\ \ \mathbf{x}_1 = (1,1,1,1),\ \ \mathbf{x}_2 = (1,0,3,0).$

$$\begin{cases} \alpha(\mathbf{x}_1\cdot\mathbf{x}_1) + \beta(\mathbf{x}_2\cdot\mathbf{x}_1) = \mathbf{y}\cdot\mathbf{x}_1 \\ \alpha(\mathbf{x}_1\cdot\mathbf{x}_2) + \beta(\mathbf{x}_2\cdot\mathbf{x}_2) = \mathbf{y}\cdot\mathbf{x}_2 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 4\alpha + 4\beta = 1 \\ 4\alpha + 10\beta = 1 \end{cases} \Longleftrightarrow \begin{cases} \alpha = 1/4 \\ \beta = 0 \end{cases}$$

$$\mathbf{p} = \tfrac{1}{4}\mathbf{x}_1 = \tfrac{1}{4}(1,1,1,1)$$
$$\mathbf{o} = \mathbf{y} - \mathbf{p} = \tfrac{1}{4}(3,-1,-1,-1)$$
$$\|\mathbf{o}\| = \tfrac{\sqrt{3}}{2},\ \ \|\mathbf{p}\| = \tfrac{1}{2}.$$

Thus the vector $\mathbf{y}$ lies at distance $\sqrt{3}/2$ from the subspace V and at distance $1/2$ from the subspace $V^{\perp}$.

Problem 5 (extra). Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.

(i) Find an orthonormal basis for V.

First we apply the Gram-Schmidt orthogonalization process to vectors $\mathbf{x}_1, \mathbf{x}_2$ and obtain an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2$ for the subspace V:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1,1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1,0,3,0) - \frac{4}{4}(1,1,1,1) = (0,-1,2,-1).$$

Then we normalize vectors $\mathbf{v}_1, \mathbf{v}_2$ to obtain an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2$ for V:

$$\|\mathbf{v}_1\| = 2 \implies \mathbf{w}_1 = \tfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \tfrac{1}{2}(1,1,1,1)$$

$$\|\mathbf{v}_2\| = \sqrt{6} \implies \mathbf{w}_2 = \tfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \tfrac{1}{\sqrt{6}}(0,-1,2,-1)$$

Problem 5 (extra). Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.

(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Since the subspace V is spanned by vectors $(1,1,1,1)$ and $(1,0,3,0)$, it is the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Then the orthogonal complement $V^\perp$ is the nullspace of A. To find the nullspace, we convert the matrix A to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

Hence a vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ belongs to $V^\perp$ if and only if

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \quad \begin{cases} x_1 + 3x_3 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{cases} \quad \iff \quad \begin{cases} x_1 = -3x_3 \\ x_2 = 2x_3 - x_4 \end{cases}$$

The general solution of the system is $(x_1, x_2, x_3, x_4) =$
$= (-3t, 2t - s, t, s) = t(-3, 2, 1, 0) + s(0, -1, 0, 1)$, where $t, s \in \mathbb{R}$.

It follows that $V^\perp$ is spanned by vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$.

The vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$ form a basis for the subspace $V^\perp$.

It remains to orthogonalize and normalize this basis:

$$\mathbf{v}_3 = \mathbf{x}_3 = (0, -1, 0, 1),$$

$$\mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\,\mathbf{v}_3 = (-3, 2, 1, 0) - \frac{-2}{2}(0, -1, 0, 1)$$

$$= (-3, 1, 1, 1),$$

$$\|\mathbf{v}_3\| = \sqrt{2} \implies \mathbf{w}_3 = \tfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \tfrac{1}{\sqrt{2}}(0, -1, 0, 1),$$

$$\|\mathbf{v}_4\| = \sqrt{12} = 2\sqrt{3} \implies \mathbf{w}_4 = \tfrac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \tfrac{1}{2\sqrt{3}}(-3, 1, 1, 1).$$

Thus the vectors $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$ and $\mathbf{w}_4 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$ form an orthonormal basis for $V^\perp$.

Problem 5 (extra). Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.

(iii) Find the distance from the vector $\mathbf{y} = (1,0,0,0)$ to the subspaces V and $V^\perp$.

For any vector $\mathbf{y} \in \mathbb{R}^4$ the orthogonal projection of $\mathbf{y}$ onto the subspace V is $\mathbf{p} = (\mathbf{y}\cdot\mathbf{w}_1)\mathbf{w}_1 + (\mathbf{y}\cdot\mathbf{w}_2)\mathbf{w}_2$ and the orthogonal projection of $\mathbf{y}$ onto $V^\perp$ is $\mathbf{o} = (\mathbf{y}\cdot\mathbf{w}_3)\mathbf{w}_3 + (\mathbf{y}\cdot\mathbf{w}_4)\mathbf{w}_4$.

Then the distance from $\mathbf{y}$ to V is $\|\mathbf{y}-\mathbf{p}\| = \|\mathbf{o}\|$ and the distance from $\mathbf{y}$ to $V^\perp$ is $\|\mathbf{y}-\mathbf{o}\| = \|\mathbf{p}\|$.

In the case $\mathbf{y} = (1,0,0,0)$, we obtain

$$\mathbf{p} = \tfrac{1}{2}\cdot\tfrac{1}{2}(1,1,1,1) = \tfrac{1}{4}(1,1,1,1),$$

$$\mathbf{o} = \tfrac{-3}{2\sqrt{3}}\cdot\tfrac{1}{2\sqrt{3}}(-3,1,1,1) = \tfrac{1}{4}(3,-1,-1,-1).$$

Hence $\|\mathbf{o}\| = \frac{\sqrt{3}}{2}$ and $\|\mathbf{p}\| = \frac{1}{2}$.