MATH 311

Topics in Applied Mathematics I

Lecture 23:
Area and volume (continued).
Multiple integrals.

Let $\mathcal{P}$ be the smallest collection of subsets of $\mathbb{R}^2$ such that it contains all polygons and if $X, Y \in \mathcal{P}$, then $X \cup Y, X \cap Y, X \setminus Y \in \mathcal{P}$.

Theorem There exists a unique function $\mu : \mathcal{P} \to \mathbb{R}$ (called the **area function**) that satisfies the following conditions:

- **(positivity)** $\mu(X) \geq 0$ for all $X \in \mathcal{P}$;
- **(additivity)** $\mu(X \cup Y) = \mu(X) + \mu(Y)$ if $X \cap Y = \emptyset$;
- **(translation invariance)** $\mu(X + \mathbf{v}) = \mu(X)$ for all $X \in \mathcal{P}$ and $\mathbf{v} \in \mathbb{R}^2$;
- $\mu(Q) = 1$, where $Q = [0,1] \times [0,1]$ is the unit square.

The area function satisfies an extra condition:

- **(monotonicity)** $\mu(X) \leq \mu(Y)$ whenever $X \subset Y$.

Now for any bounded set $X \subset \mathbb{R}^2$ we let $\overline{\mu}(X) = \inf\limits_{X \subset Y} \mu(Y)$ and $\underline{\mu}(X) = \sup\limits_{Z \subset X} \mu(Z)$. Note that $\underline{\mu}(X) \leq \overline{\mu}(X)$. In the case of equality, the set X is called **Jordan measurable** and we let $\mathrm{area}(X) = \overline{\mu}(X)$.

Area, volume, and determinants

- 2×2 determinants and plane geometry

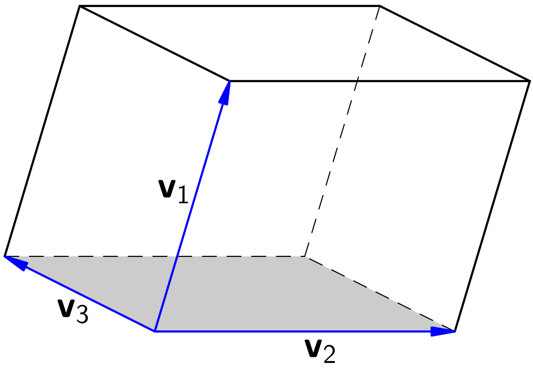
Let P be a parallelogram in the plane $\mathbb{R}^2$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are represented by adjacent sides of P. Then $\mathrm{area}(P) = |\det A|$, where $A = (\mathbf{v}_1, \mathbf{v}_2)$, a matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$.

Consider a linear operator $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L_A(\mathbf{v}) = A\mathbf{v}$ for any column vector $\mathbf{v}$. Then $\mathrm{area}(L_A(D)) = |\det A| \, \mathrm{area}(D)$ for any bounded domain D.

- 3×3 determinants and space geometry

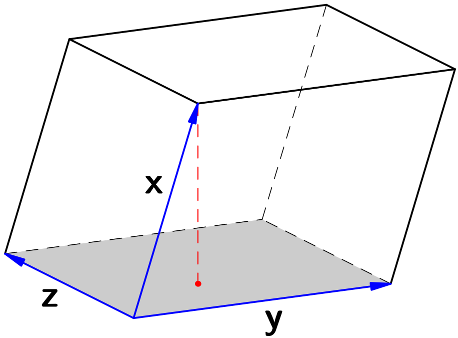
Let Π be a parallelepiped in space $\mathbb{R}^3$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are represented by adjacent edges of Π. Then $\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, a matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Similarly, $\mathrm{volume}(L_B(D)) = |\det B| \, \mathrm{volume}(D)$ for any bounded domain $D \subset \mathbb{R}^3$.



$\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Note that the parallelepiped Π is the image under L_B of a unit cube whose adjacent edges are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the right-hand rule. We say that L_B **preserves orientation** if it preserves the hand rule for any basis. This is the case if and only if $\det B > 0$.

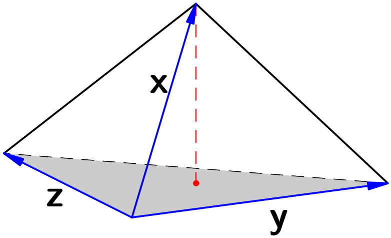


Parallelepiped is a prism.

(Volume) $=$ (area of the base) $\times$ (height)

Area of the base $= \|\mathbf{y} \times \mathbf{z}\|$

Volume $= |\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$



Tetrahedron is a pyramid.

$$(\text{Volume}) = \frac{1}{3}(\text{area of the base}) \times (\text{height})$$

$$\text{Area of the base} = \frac{1}{2}\|\mathbf{y} \times \mathbf{z}\|$$

$$\implies \quad \text{Volume} = \frac{1}{6}|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$$

Riemann sums in two dimensions

Consider a closed coordinate rectangle
$R = [a, b] \times [c, d] \subset \mathbb{R}^2$.

Definition. A **Riemann sum** of a function $f : R \to \mathbb{R}$ with respect to a partition $P = \{D_1, D_2, \dots, D_n\}$ of R generated by samples $t_j \in D_j$ is a sum

$$\mathcal{S}(f, P, t_j) = \sum\nolimits_{j=1}^{n} f(t_j) \operatorname{area}(D_j).$$

The norm of the partition P is $\|P\| = \max_{1 \le j \le n} \operatorname{diam}(D_j)$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

If this is the case, then the function f is called **integrable** on R and the limit $I(f)$ is called the **integral** of f over R.

Double integral

Closed coordinate rectangle $R = [a, b] \times [c, d]$
$= \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \ \ c \leq y \leq d\}$.

Notation: $\displaystyle\iint_R f \, dA$ or $\displaystyle\iint_R f(x, y) \, dx \, dy$.

Theorem 1 If f is continuous on the closed rectangle R, then f is integrable.

Theorem 2 A function $f : R \to \mathbb{R}$ is Riemann integrable on the rectangle R if and only if f is bounded on R and continuous almost everywhere on R (that is, the set of discontinuities of f has zero area).

Fubini's Theorem

Fubini's Theorem allows us to reduce a multiple integral to a repeated one-dimensional integral.

Theorem If a function f is integrable on $R = [a, b] \times [c, d]$, then

$$\iint_R f \, dA = \int_a^b \Big(\int_c^d f(x, y) \, dy \Big) \, dx = \int_c^d \Big(\int_a^b f(x, y) \, dx \Big) \, dy.$$

In particular, this implies that we can change the order of integration in a repeated integral.

Corollary If a function g is integrable on $[a, b]$ and a function h is integrable on $[c, d]$, then the function $f(x, y) = g(x)h(y)$ is integrable on $R = [a, b] \times [c, d]$ and

$$\iint_R g(x)h(y) \, dx \, dy = \int_a^b g(x) \, dx \cdot \int_c^d h(y) \, dy.$$

Integrals over general domains

Suppose $f : D \to \mathbb{R}$ is a function defined on a (Jordan) measurable set $D \subset \mathbb{R}^2$. Since D is bounded, it is contained in a rectangle R. To define the integral of f over D, we extend the function f to a function on R:

$$f^{\text{ext}}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D, \\ 0 & \text{if } (x,y) \notin D. \end{cases}$$

Definition. $\displaystyle\iint_D f\,dA$ is defined to be $\displaystyle\iint_R f^{\text{ext}}\,dA$.

In particular, $\displaystyle \mathrm{area}(D) = \iint_D 1\,dA$.

Integration as a linear operation

Theorem 1 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, then the sum $f + g$ is also integrable on D and

$$\iint_D (f + g)\, dA = \iint_D f\, dA + \iint_D g\, dA.$$

Theorem 2 If a function f is integrable on a set $D \subset \mathbb{R}^2$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on D and

$$\iint_D \alpha f\, dA = \alpha \iint_D f\, dA.$$

More properties of integrals

Theorem 3 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, and $f(x,y) \leq g(x,y)$ for all $(x,y) \in D$, then

$$\iint_D f\,dA \leq \iint_D g\,dA.$$

Theorem 4 If a function f is integrable on sets $D_1, D_2 \subset \mathbb{R}^2$, then it is integrable on their union $D_1 \cup D_2$. Moreover, if the sets D_1 and D_2 are disjoint up to a set of zero area, then

$$\iint_{D_1 \cup D_2} f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA.$$

Change of variables in a double integral

Theorem Let $D \subset \mathbb{R}^2$ be a measurable domain and f be an integrable function on D. If $\mathbf{T} = (u, v)$ is a smooth coordinate mapping such that $\mathbf{T}^{-1}$ is defined on D, then

$$\iint_D f(u, v)\, du\, dv$$

$$= \iint_{\mathbf{T}^{-1}(D)} f\big(u(x, y), v(x, y)\big) \left|\det \frac{\partial(u, v)}{\partial(x, y)}\right| \, dx\, dy.$$

In particular, the integral in the right-hand side is well defined.

Problem Evaluate a double integral

$$\iint_D f(x,y)\,dx\,dy$$

over a disc D bounded by the circle $(x-x_0)^2+(y-y_0)^2=R^2$.

To evaluate the integral, we move the origin to (x_0,y_0) and then switch to polar coordinates (r,ϕ). That is, we use the substitution $(x,y)=T(r,\phi)=(x_0+r\cos\phi,\ y_0+r\sin\phi)$.

Jacobian matrix: $J=\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi}\end{pmatrix}=\begin{pmatrix}\cos\phi & -r\sin\phi\\ \sin\phi & r\cos\phi\end{pmatrix}$.

Then $\det J=r\cos^2\phi+r\sin^2\phi=r$. Hence

$$\iint_D f(x,y)\,dx\,dy=\iint_{T^{-1}(D)} f(x_0+r\cos\phi,\ y_0+r\sin\phi)\,|\det J|\,dr\,d\phi$$

$$=\int_0^{2\pi}\int_0^R f(x_0+r\cos\phi,\ y_0+r\sin\phi)\,r\,dr\,d\phi.$$

Problem Evaluate a double integral

$$\iint_P f(x,y)\,dx\,dy$$

over a parallelogram P with vertices $(-1,-1)$, $(1,0)$, $(2,2)$, and $(0,1)$.

Adjacent edges of the parallelogram P are represented by vectors $\mathbf{v}_1 = (1,0)-(-1,-1) = (2,1)$ and $\mathbf{v}_2 = (0,1)-(-1,-1) = (1,2)$.

Consider a transformation L of the plane $\mathbb{R}^2$ given by

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2u+v-1 \\ u+2v-1 \end{pmatrix}$$

(columns of the matrix are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$). By construction, L maps the unit square $[0,1]\times[0,1]$ onto the parallelogram P. The Jacobian matrix J of L is the same at any point: $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Changing coordinates in the integral from (x, y) to (u, v) so that

$$(x, y) = L(u, v) = (2u + v - 1,\ u + 2v - 1),$$

we obtain

$$\iint_P f(x, y)\, dx\, dy$$

$$= \iint_{L^{-1}(P)} f(2u + v - 1,\ u + 2v - 1)\, |\det J|\, du\, dv$$

$$= 3 \int_0^1 \int_0^1 f(2u + v - 1,\ u + 2v - 1)\, du\, dv.$$

Triple integral

To integrate in $\mathbb{R}^3$, volumes are used instead of areas in $\mathbb{R}^2$. Instead of coordinate rectangles, basic sets are coordinate boxes (or bricks) $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$. Then we can define an integral of a function f over a measurable set $D \subset \mathbb{R}^3$.

Notation: $\iiint_D f\,dV$ or $\iiint_D f(x,y,z)\,dx\,dy\,dz$.

The properties of triple integrals are completely analogous to those of double integrals. In particular, Fubini's Theorem is formulated as follows.

Theorem If a function f is integrable on a brick $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$, then

$$\iiint_B f\,dV = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \Big(\int_{a_3}^{b_3} f(x,y,z)\,dz \Big)\,dy \right) dx.$$

Problem Find the volume of a tetrahedron (i.e., triangular pyramid) with vertices at points $(0,2,1)$, $(1,0,0)$, $(2,1,2)$, and $(3,1,1)$.

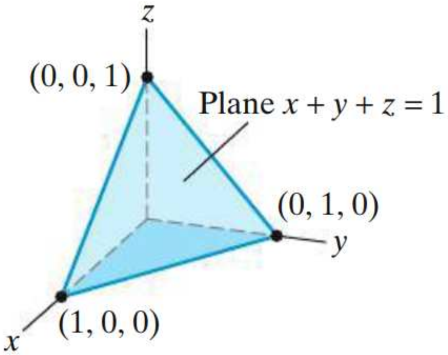
Let P denote the pyramid. Let $A_0 = (1,0,0)$, $A_1 = (0,2,1)$, $A_2 = (2,1,2)$ and $A_3 = (3,1,1)$. Three edges adjacent to A_0 are represented by vectors

$$\mathbf{v}_1 = \overrightarrow{A_0A_1} = (0,2,1) - (1,0,0) = (-1,2,1),$$
$$\mathbf{v}_2 = \overrightarrow{A_0A_2} = (2,1,2) - (1,0,0) = (1,1,2),$$
$$\mathbf{v}_3 = \overrightarrow{A_0A_3} = (3,1,1) - (1,0,0) = (2,1,1).$$

Consider a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The matrix is $M = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.



z
(0, 0, 1)
Plane x + y + z = 1
(0, 1, 0)
y
(1, 0, 0)
x

By construction, $T(0,0,0) = A_0$, $T(1,0,0) = A_1$, $T(0,1,0) = A_2$ and $T(0,0,1) = A_3$. It follows that $T^{-1}(P)$ is the triangular pyramid with vertices at points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Consider $(0,0,1)$ to be the apex of the pyramid $T^{-1}(P)$. Then the base is an isosceles right triangle with legs of length 1. Its area equals $\frac{1}{2}$. Besides, the edge $(0,0,0)$–$(0,0,1)$ is the altitude. Therefore the volume of the pyramid $T^{-1}(P)$ equals $\frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$.

We have $\mathrm{volume}(T(D)) = |\det M| \, \mathrm{volume}(D)$ for any domain $D \subset \mathbb{R}^3$. In particular, $\mathrm{volume}(P) = |\det M|/6$.

$$\det M = \begin{vmatrix} -1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{vmatrix} = 6.$$

Thus $\mathrm{volume}(P) = 6 \cdot \frac{1}{6} = 1$.