Lecture 6:
Matrix algebra (continued).
Determinants.
Theorem 1  Any elementary row operation \( \sigma \) on matrices with \( n \) rows can be simulated as left multiplication by a certain \( n \times n \) matrix \( E_\sigma \) (called an elementary matrix).

Theorem 2  Elementary matrices are invertible.

Proof: Suppose \( E_\sigma \) is an \( n \times n \) elementary matrix corresponding to an operation \( \sigma \). We know that \( \sigma \) can be undone by another elementary row operation \( \tau \). It is easy to check that \( \sigma \) undoes \( \tau \) as well. Then for any matrix \( A \) with \( n \) rows we have \( E_\tau E_\sigma A = A \) (since \( \tau \) undoes \( \sigma \)) and \( E_\sigma E_\tau A = A \) (since \( \sigma \) undoes \( \tau \)). In particular, \( E_\tau E_\sigma I = E_\sigma E_\tau I = I \), which implies that \( E_\tau = E_\sigma^{-1} \).

Theorem 3  A square matrix is invertible if and only if it can be expanded into a product of elementary matrices.
**Theorem**  Suppose that a sequence of elementary row operations converts a matrix $A$ into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix $A^{-1}$.

**Proof:** Let $E_1, E_2, \ldots, E_k$ be elementary matrices that correspond to elementary row operations converting $A$ into $I$. Then $E_k E_{k-1} \ldots E_2 E_1 A = I$.

Applying the same sequence of operations to the identity matrix $I$, we obtain the matrix

$$B = E_k E_{k-1} \ldots E_2 E_1 I = E_k E_{k-1} \ldots E_2 E_1.$$

Therefore $BA = I$. Besides, $B$ is invertible since elementary matrices are invertible. Then $B^{-1}(BA) = B^{-1}I$. It follows that $A = B^{-1}$, hence $B = A^{-1}$. 
**Theorem**  A square matrix $A$ is invertible if and only if $x = 0$ is the only solution of the matrix equation $Ax = 0$.

**Corollary 1**  For any $n \times n$ matrices $A$ and $B$,

\[ BA = I \iff AB = I. \]

**Proof:**  It is enough to prove that $BA = I \implies AB = I$.

Assume $BA = I$. Then $Ax = 0 \implies B(Ax) = B0$

\[ \implies (BA)x = 0 \implies x = 0. \]  By the theorem, $A$ is invertible.

Then $BA = I \implies A(BA)A^{-1} = AIA^{-1} \implies AB = I$.

**Corollary 2**  Suppose $A$ and $B$ are $n \times n$ matrices. If the product $AB$ is invertible, then both $A$ and $B$ are invertible.

**Proof:**  Let $C = B(AB)^{-1}$ and $D = (AB)^{-1}A$. Then

\[ AC = A(B(AB)^{-1}) = (AB)(AB)^{-1} = I \quad \text{and} \]

\[ DB = ((AB)^{-1}A)B = (AB)^{-1}(AB) = I. \]  By Corollary 1, $C = A^{-1}$ and $D = B^{-1}$. 

Transposition of a matrix

**Definition.** Given a matrix $A$, the **transpose** of $A$, denoted $A^T$, is the matrix whose rows are columns of $A$ (and whose columns are rows of $A$). That is, if $A = (a_{ij})$ then $A^T = (b_{ij})$, where $b_{ij} = a_{ji}$.

**Examples.**

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}^T = \begin{pmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{pmatrix},
\]

\[
\begin{pmatrix}
7 \\
8 \\
9
\end{pmatrix}^T = (7, 8, 9),
\]

\[
\begin{pmatrix}
4 & 7 \\
7 & 0
\end{pmatrix}^T = \begin{pmatrix}
4 & 7 \\
7 & 0
\end{pmatrix}.
\]
Properties of transposes:

- \((A^T)^T = A\)
- \((A + B)^T = A^T + B^T\)
- \((rA)^T = rA^T\)
- \((AB)^T = B^T A^T\)
- \((A_1 A_2 \ldots A_k)^T = A_k^T \ldots A_2^T A_1^T\)
- \((A^{-1})^T = (A^T)^{-1}\)
Definition. A square matrix $A$ is said to be symmetric if $A^T = A$.

For example, any diagonal matrix is symmetric.

**Proposition** For any square matrix $A$ the matrices $B = AA^T$ and $C = A + A^T$ are symmetric.

**Proof:**

$$B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B,$$

$$C^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = C.$$
Determinants

**Determinant** is a scalar assigned to each square matrix.

**Notation.** The determinant of a matrix

\[ A = (a_{ij})_{1 \leq i,j \leq n} \]

is denoted \( \det A \) or

\[
\begin{vmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{vmatrix}
\]

**Principal property:** \( \det A \neq 0 \) if and only if a system of linear equations with the coefficient matrix \( A \) has a unique solution. Equivalently, \( \det A \neq 0 \) if and only if the matrix \( A \) is invertible.
Definition in low dimensions

Definition. \( \det(a) = a \),
\[
\begin{vmatrix}
  a & b \\
  c & d
\end{vmatrix} = ad - bc,
\]
\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -
\]
\[
-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.
\]

\[+::\]
\[
\begin{pmatrix}
  * & * & * \\
  * & * & * \\
  * & * & *
\end{pmatrix},
\begin{pmatrix}
  * & * & * \\
  * & * & * \\
  * & * & *
\end{pmatrix},
\begin{pmatrix}
  * & * & * \\
  * & * & * \\
  * & * & *
\end{pmatrix}.
\]

\[\:-::\]
\[
\begin{pmatrix}
  * & * & * \\
  * & * & * \\
  * & * & *
\end{pmatrix},
\begin{pmatrix}
  * & * & * \\
  * & * & * \\
  * & * & *
\end{pmatrix},
\begin{pmatrix}
  * & * & * \\
  * & * & * \\
  * & * & *
\end{pmatrix}.
\]
Examples: $2 \times 2$ matrices

\[
\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12,
\]

\[
\begin{vmatrix} -2 & 5 \\ 0 & 3 \end{vmatrix} = -6, \quad \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14,
\]

\[
\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} = 0,
\]

\[
\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 1 \\ 8 & 4 \end{vmatrix} = 0.
\]
Examples: $3 \times 3$ matrices

\[
\begin{vmatrix}
3 & -2 & 0 \\
1 & 0 & 1 \\
-2 & 3 & 0
\end{vmatrix}
= 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 - \\
0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3
= 4 - 9 = -5,
\]

\[
\begin{vmatrix}
1 & 4 & 6 \\
0 & 2 & 5 \\
0 & 0 & 3
\end{vmatrix}
= 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0 - \\
-6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0
= 1 \cdot 2 \cdot 3 = 6.
\]
General definition

The general definition of the determinant is quite complicated as there is no simple explicit formula.

There are several approaches to defining determinants.

**Approach 1 (original):** an explicit (but very complicated) formula.

**Approach 2 (axiomatic):** we formulate properties that the determinant should have.

**Approach 3 (inductive):** the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n - 1) \times (n - 1)$ matrices.
Classical definition

**Definition.** If \( A = (a_{ij}) \) is an \( n \times n \) matrix then

\[
\det A = \sum_{\pi \in S_n} \text{sgn}(\pi) \ a_{1,\pi(1)} \ a_{2,\pi(2)} \ldots a_{n,\pi(n)},
\]

where \( \pi \) runs over \( S_n \), the set of all permutations of \( \{1, 2, \ldots, n\} \), and \( \text{sgn}(\pi) \) denotes the sign of the permutation \( \pi \).

**Remarks.**

- A **permutation** of the set \( \{1, 2, \ldots, n\} \) is an invertible mapping of this set onto itself. There are \( n! \) such mappings.
- The **sign** \( \text{sgn}(\pi) \) can be 1 or \(-1\). Its definition is rather complicated.
Axiomatic definition

\( \mathcal{M}_{n,n}(\mathbb{R}) \): the set of \( n \times n \) matrices with real entries.

**Theorem**  There exists a unique function \( \text{det} : \mathcal{M}_{n,n}(\mathbb{R}) \to \mathbb{R} \) (called the determinant) with the following properties:

(\text{D1}) if a row of a matrix is multiplied by a scalar \( r \), the determinant is also multiplied by \( r \);

(\text{D2}) if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;

(\text{D3}) if we interchange two rows of a matrix, the determinant changes its sign;

(\text{D4}) \( \det I = 1 \).
Corollary 1  Suppose $A$ is a square matrix and $B$ is obtained from $A$ applying elementary row operations. Then $\det A = 0$ if and only if $\det B = 0$.

Corollary 2  $\det B = 0$ whenever the matrix $B$ has a zero row.

*Hint:* Multiply the zero row by the zero scalar.

Corollary 3  $\det A = 0$ if and only if the matrix $A$ is not invertible.

*Idea of the proof:* Let $B$ be the reduced row echelon form of $A$. If $A$ is invertible then $B = I$; otherwise $B$ has a zero row.

*Remark.* The same argument proves that properties (D1)–(D4) are enough to evaluate any determinant.
Row echelon form of a square matrix $A$:

\[
\begin{pmatrix}
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \\
\ast & \ast & \ast & \ast & & \\
\ast & \ast & & & & \\
\ast & & & & & \\
\ast & & & & & \\
\end{pmatrix}
\quad \begin{pmatrix}
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \\
\ast & \ast & \ast & \ast & & \\
\ast & \ast & & & & \\
\ast & & & & & \\
\ast & & & & & \\
\end{pmatrix}
\]

$\det A \neq 0$    $\det A = 0$
Example. \( A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \), \( \det A = ? \)

In the previous lecture, we have transformed the matrix \( A \) into the identity matrix using elementary row operations:

- interchange the 1st row with the 2nd row,
- add \(-3\) times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by \(-0.5\),
- add \(-3\) times the 2nd row to the 3rd row,
- multiply the 3rd row by \(-0.4\),
- add \(-1.5\) times the 3rd row to the 2nd row,
- add \(-1\) times the 3rd row to the 1st row.
Example. \( A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \), \( \det A = ? \)

In the previous lecture, we have transformed the matrix \( A \) into the identity matrix using elementary row operations.

These included two row multiplications, by \(-0.5\) and by \(-0.4\), and one row exchange.

It follows that

\[
\det I = - (-0.5) (-0.4) \det A = (-0.2) \det A.
\]

Hence \( \det A = -5 \det I = -5 \).