

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1. Find a quadratic polynomial $p(x)$ such that $p(-1) = p(3) = 6$ and $p'(2) = p(1)$.

Problem 2. Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}$.

- (i) Evaluate the determinant of the matrix A .
- (ii) Find the inverse matrix A^{-1} .

Problem 3. Consider a linear transformation $F : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ given by

$$F(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_3 + x_5, 2x_1 - x_2 + x_4).$$

Find a basis for the kernel of F , then extend it to a basis for \mathbb{R}^5 .

Problem 4. Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear operator on \mathbb{R}^3 such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

- (i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for \mathbb{R}^3 .
- (ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
- (iii) Find the matrix of the operator L relative to the standard basis.

Problem 5. Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

- (i) Find all eigenvalues of the matrix B .
- (ii) Find a basis for \mathbb{R}^3 consisting of eigenvectors of B .
- (iii) Find an orthonormal basis for \mathbb{R}^3 consisting of eigenvectors of B .
- (iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.

Problem 6. Prove that any subspace of the vector space \mathbb{R}^n is the solution set of some system of linear equations in n variables.

Problem 7. Let V be a subspace of \mathbb{R}^4 spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (2, 0, -1, 1)$, and $\mathbf{x}_3 = (0, 1, 1, 0)$.

- (i) Find the distance from the point $\mathbf{y} = (0, 0, 0, 4)$ to the subspace V .
- (ii) Find the distance from the point \mathbf{y} to the orthogonal complement V^\perp .

Problem 8. Consider a linear operator $K : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$K(\mathbf{x}) = C\mathbf{x}, \quad \text{where} \quad C = \frac{1}{9} \begin{pmatrix} -4 & 7 & 4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}.$$

- (i) Explain why K is a rigid motion and, specifically, a rotation about an axis.
- (ii) Find the axis of rotation.
- (iii) Find the angle of rotation.