MATH 323

Linear Algebra

Lecture 3:
Gauss-Jordan reduction (continued).
Applications of systems of linear equations.
Matrix algebra.

How to solve a system of linear equations

- Order the variables.
- Write down the augmented matrix of the system.
- Convert the matrix to **row echelon form**.
- Check for consistency.
- Convert the matrix to **reduced row echelon form**.
- Write down the system corresponding to the reduced row echelon form.
- Determine leading and free variables.
- Rewrite the system so that the leading variables are on the left while everything else is on the right.
- Assign parameters to the free variables and write down the general solution in parametric form.

Example with a parameter.

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases} \qquad (a \in \mathbb{R})$$

The system is **homogeneous** (all right-hand sides are zeros). Therefore it is consistent ($x = y = z = 0$ is a solution).

Augmented matrix: $\left(\begin{array}{ccr|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$

Since the 1st row cannot serve as a pivotal one, we interchange it with the 2nd row:

$$\left(\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$$

Now we can start the elimination.

First subtract the 1st row from the 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right)$$

The 2nd row is our new pivotal row.

Subtract the 2nd row from the 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right)$$

At this point row reduction splits into two cases.

Case 1: $a \neq 1$. In this case, multiply the 3rd row by $(a-1)^{-1}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

The matrix is converted into row echelon form.
We proceed towards reduced row echelon form.

Subtract 3 times the 3rd row from the 2nd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Add 2 times the 3rd row to the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Finally, subtract the 2nd row from the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

Thus $x = y = z = 0$ is the only solution.

Case 2: $a = 1$. In this case, the matrix is already in row echelon form:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

To get reduced row echelon form, subtract the 2nd row from the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z is a free variable.

$$\begin{cases} x - 5z = 0 \\ y + 3z = 0 \end{cases} \iff \begin{cases} x = 5z \\ y = -3z \end{cases}$$

System of linear equations:

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases}$$

Solution: If $a \neq 1$ then $(x, y, z) = (0, 0, 0)$;
if $a = 1$ then $(x, y, z) = (5t, -3t, t)$, $t \in \mathbb{R}$.

Theorem Any matrix can be converted into row echelon form by applying elementary row operations.

Sketch of the proof: The proof is by induction on the number of columns in the matrix. It relies on the next lemma.

Lemma Any matrix can be converted to one of the following forms using elementary row operations: **(i)** $(1\ a_{12}\ a_{13}\ \dots\ a_{1n})$;

$$\textbf{(ii)}\ \left(\begin{array}{c|ccc} 1 & a_{12} & \dots & a_{1n} \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right);\ \textbf{(iii)}\ \left(\begin{array}{c} 1 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right);\ \textbf{(iv)}\ \left(\begin{array}{c|c} 0 & \\ \vdots & B \\ 0 & \end{array}\right);\ \textbf{(v)}\ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

In the cases (i), (iii) and (v), we already have a row echelon form. In the cases (ii) and (iv), it is enough to convert the matrix B to row echelon form. Moreover, the row reduction on the block B can be simulated by applying elementary row operations to the entire matrix.

Properties of row echelon form

Let C be a matrix in the row echelon form (resp. reduced row echelon form). We say that C is a **row echelon form** (resp. **reduced row echelon form**) of a matrix A if C can be obtained from A by applying elementary row operations.

Theorem 1 For any matrix, the reduced row echelon form exists and is unique.

Theorem 2 Suppose A and B are matrices of the same dimensions. Then the following conditions are equivalent:

(i) A and B share a reduced row echelon form;
(ii) A and B share a row echelon form;
(iii) A can be obtained from B by applying elementary row operations.

Applications of systems of linear equations

Problem 1. Find the point of intersection of the lines $x - y = -2$ and $2x + 3y = 6$ in $\mathbb{R}^2$.

$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases}$$

Problem 2. Find the point of intersection of the planes $x - y = 2$, $2x - y - z = 3$, and $x + y + z = 6$ in $\mathbb{R}^3$.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

Method of undetermined coefficients often involves solving systems of linear equations.

Problem 3. Find a quadratic polynomial $p(x)$ such that $p(1) = 4$, $p(2) = 3$, and $p(3) = 4$.

Suppose that $p(x) = ax^2 + bx + c$. Then $p(1) = a + b + c$, $p(2) = 4a + 2b + c$, $p(3) = 9a + 3b + c$.

$$\begin{cases} a + b + c = 4 \\ 4a + 2b + c = 3 \\ 9a + 3b + c = 4 \end{cases}$$

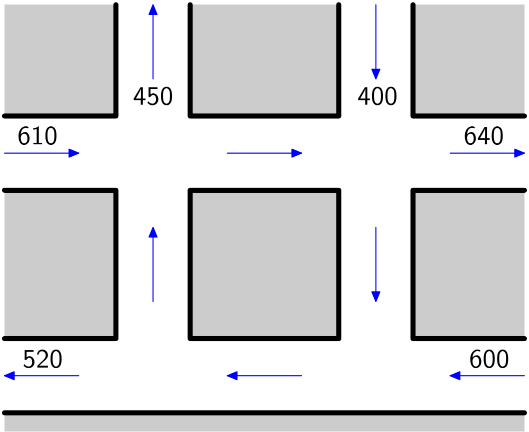
Method of undetermined coefficients often involves solving systems of linear equations.

Problem 3. Find a quadratic polynomial $p(x)$ such that $p(1) = 4$, $p(2) = 3$, and $p(3) = 4$.

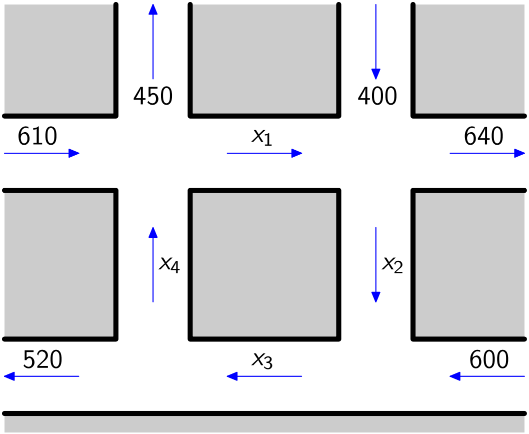
Alternative choice of coefficients: $p(x) = \tilde{a} + \tilde{b}x + \tilde{c}x^2$. Then $p(1) = \tilde{a} + \tilde{b} + \tilde{c}$, $p(2) = \tilde{a} + 2\tilde{b} + 4\tilde{c}$, $p(3) = \tilde{a} + 3\tilde{b} + 9\tilde{c}$.

$$\begin{cases} \tilde{a} + \tilde{b} + \tilde{c} = 4 \\ \tilde{a} + 2\tilde{b} + 4\tilde{c} = 3 \\ \tilde{a} + 3\tilde{b} + 9\tilde{c} = 4 \end{cases}$$

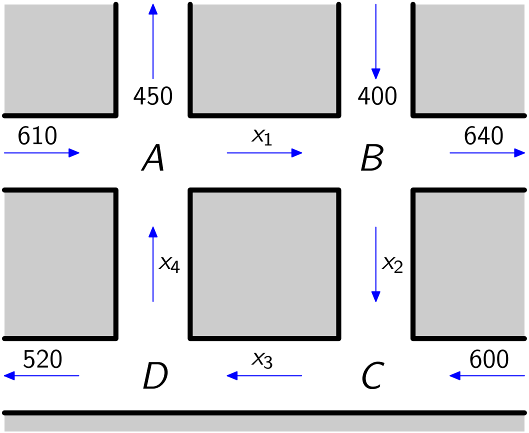
Traffic flow



Problem. Determine the amount of traffic between each of the four intersections.



$$x_1 = ?, \quad x_2 = ?, \quad x_3 = ?, \quad x_4 = ?$$



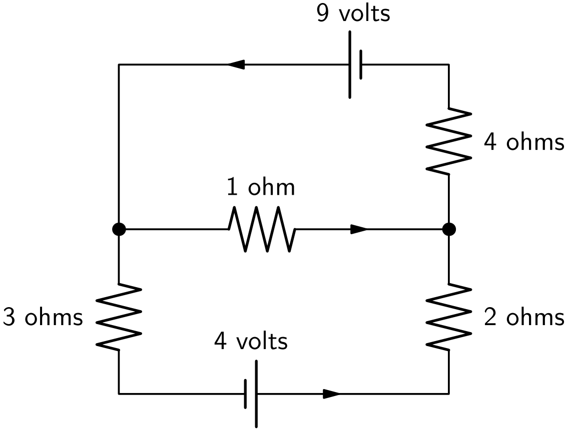
At each intersection, the incoming traffic has to match the outgoing traffic.

Intersection A: $x_4 + 610 = x_1 + 450$
Intersection B: $x_1 + 400 = x_2 + 640$
Intersection C: $x_2 + 600 = x_3$
Intersection D: $x_3 = x_4 + 520$

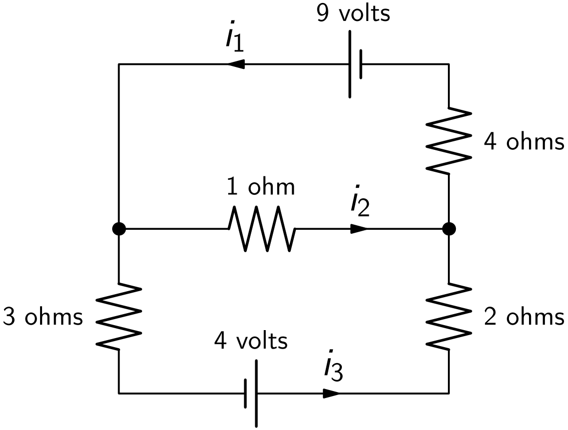
$$\begin{cases} x_4 + 610 = x_1 + 450 \\ x_1 + 400 = x_2 + 640 \\ x_2 + 600 = x_3 \\ x_3 = x_4 + 520 \end{cases}$$

$$\iff \begin{cases} -x_1 + x_4 = -160 \\ x_1 - x_2 = 240 \\ x_2 - x_3 = -600 \\ x_3 - x_4 = 520 \end{cases}$$

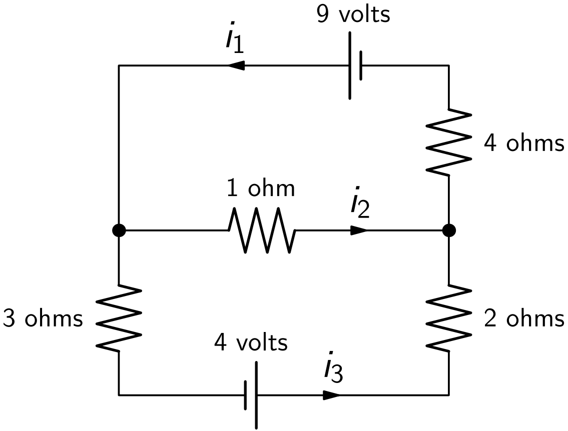
Electrical network



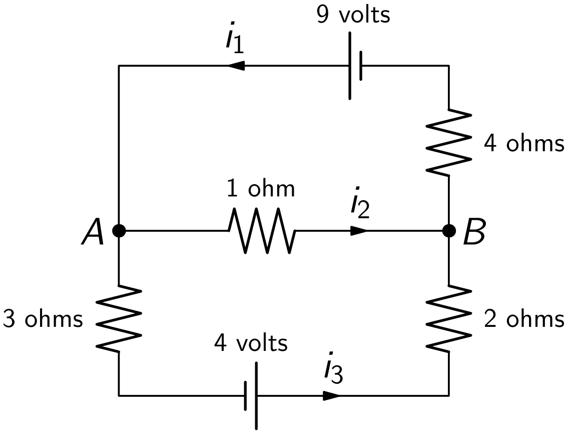
Problem. Determine the amount of current in each branch of the network.



$$i_1 = ?, \quad i_2 = ?, \quad i_3 = ?$$



Kirchhof's law #1 (junction rule): at every node the sum of the incoming currents equals the sum of the outgoing currents.



Node A: $\quad i_1 = i_2 + i_3$

Node B: $\quad i_2 + i_3 = i_1$

Electrical network

Kirchhof's law #2 (loop rule): around every loop the algebraic sum of all voltages is zero.

Ohm's law: for every resistor the voltage drop E, the current i, and the resistance R satisfy $E = iR$.

$$\begin{aligned} \text{Top loop:} \quad & 9 - i_2 - 4i_1 = 0 \\ \text{Bottom loop:} \quad & 4 - 2i_3 + i_2 - 3i_3 = 0 \\ \text{Big loop:} \quad & 4 - 2i_3 - 4i_1 + 9 - 3i_3 = 0 \end{aligned}$$

Remark. The 3rd equation is the sum of the first two equations.

$$\begin{cases} i_1 = i_2 + i_3 \\ 9 - i_2 - 4i_1 = 0 \\ 4 - 2i_3 + i_2 - 3i_3 = 0 \end{cases}$$

$$\iff \begin{cases} i_1 - i_2 - i_3 = 0 \\ 4i_1 + i_2 = 9 \\ -i_2 + 5i_3 = 4 \end{cases}$$

Matrices (revisited)

Definition. An **m-by-n matrix** is a rectangular array of numbers that has m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notation: $A = (a_{ij})_{1\le i\le n,\, 1\le j\le m}$ or simply $A = (a_{ij})$ if the dimensions are known.

An n-dimensional vector can be represented as a $1 \times n$ matrix (row vector) or as an $n \times 1$ matrix (column vector):

$$(x_1, x_2, \ldots, x_n) \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

An $m \times n$ matrix $A = (a_{ij})$ can be regarded as a column of n-dimensional row vectors or as a row of m-dimensional column vectors:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}, \qquad \mathbf{v}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$$

$$A = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n), \qquad \mathbf{w}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Vector algebra

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be n-dimensional vectors, and $r \in \mathbb{R}$ be a scalar.

Vector sum: $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

Scalar multiple: $r\mathbf{a} = (ra_1, ra_2, \ldots, ra_n)$

Zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$

Negative of a vector: $-\mathbf{b} = (-b_1, -b_2, \ldots, -b_n)$

Vector difference:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$$

Given n-dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and scalars $r_1, r_2, \ldots, r_k$, the expression

$$\boxed{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k}$$

is called a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Also, *vector addition* and *scalar multiplication* are called **linear operations**.

Matrix algebra

Definition. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. The **sum** $A + B$ is defined to be the $m \times n$ matrix $C = (c_{ij})$ such that $\boxed{c_{ij} = a_{ij} + b_{ij}}$ for all indices i, j.

That is, two matrices with the same dimensions can be added by adding their corresponding entries.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{pmatrix}$$

Definition. Given an $m \times n$ matrix $A = (a_{ij})$ and a number r, the **scalar multiple** rA is defined to be the $m \times n$ matrix $D = (d_{ij})$ such that $\boxed{d_{ij} = ra_{ij}}$ for all indices i, j.

That is, to multiply a matrix by a scalar r, one multiplies each entry of the matrix by r.

$$r \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{pmatrix}$$

The $m \times n$ **zero matrix** (all entries are zeros) is denoted O_{mn} or simply O.

Negative of a matrix: $-A$ is defined as $(-1)A$.

Matrix **difference**: $A - B$ is defined as $A + (-B)$.

As far as the *linear operations* (addition and scalar multiplication) are concerned, the $m \times n$ matrices can be regarded as *mn*-dimensional vectors.

Examples

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} 5 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \qquad A - B = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$2C = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \qquad 3D = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix},$$

$$2C + 3D = \begin{pmatrix} 7 & 3 \\ 0 & 5 \end{pmatrix},$$

$A + D$ is not defined.

Properties of linear operations

$$(A + B) + C = A + (B + C)$$

$$A + B = B + A$$

$$A + O = O + A = A$$

$$A + (-A) = (-A) + A = O$$

$$r(sA) = (rs)A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$1A = A$$

$$0A = O$$

$$(-1)A = -A$$