MATH 323

Linear Algebra

Lecture 8a: Determinants (continued).

More properties of determinants

Determinants and matrix multiplication:

- if A and B are $n \times n$ matrices then
$$\det(AB) = \det A \cdot \det B,$$
- if A and B are $n \times n$ matrices then
$$\det(AB) = \det(BA),$$
- if A is an invertible matrix then
$$\det(A^{-1}) = (\det A)^{-1}.$$

Determinants and scalar multiplication:

- if A is an $n \times n$ matrix and $r \in \mathbb{R}$ then
$$\det(rA) = r^n \det A.$$

Examples

$$X = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -2 & 1 \end{pmatrix}.$$

$$\det X = (-1)\cdot 2\cdot(-3) = 6, \quad \det Y = \det Y^T = 3,$$

$$\det(XY) = 6\cdot 3 = 18, \quad \det(YX) = 3\cdot 6 = 18,$$

$$\det(Y^{-1}) = 1/3, \quad \det(XY^{-1}) = 6/3 = 2,$$

$$\det(XYX^{-1}) = \det Y = 3, \quad \det(X^{-1}Y^{-1}XY) = 1,$$

$$\det(2X) = 2^3 \det X = 2^3\cdot 6 = 48,$$

$$\det(-3X^TXY^{-4}) = (-3)^3\cdot 6\cdot 6\cdot 3^{-4} = -12.$$

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1)\times(n-1)$ submatrix obtained by deleting the ith row and the jth column of A.

Theorem For any $1 \leq k, m \leq n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj},$$

(*expansion by kth row*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}.$$

(*expansion by mth column*)

Determinants and the inverse matrix

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1)\times(n-1)$ submatrix obtained by deleting the ith row and the jth column of A. The **cofactor matrix** of A is an $n \times n$ matrix $\widetilde{A} = (\alpha_{ij})$ defined by $\alpha_{ij} = (-1)^{i+j} \det M_{ij}$.

Theorem $\widetilde{A}^T A = A\,\widetilde{A}^T = (\det A)I$.

Sketch of the proof: $A\,\widetilde{A}^T = (\det A)I$ means that

$$\sum_{j=1}^{n}(-1)^{k+j} a_{kj} \det M_{kj} = \det A \quad \text{for all } k,$$

$$\sum_{j=1}^{n}(-1)^{k+j} a_{mj} \det M_{kj} = 0 \quad \text{for } m \neq k.$$

Indeed, the 1st equality is the expansion of $\det A$ by the kth row. The 2nd equality is an analogous expansion of $\det B$, where the matrix B is obtained from A by replacing its kth row with a copy of the mth row (clearly, $\det B = 0$). $\widetilde{A}^T A = (\det A)I$ is verified similarly, using column expansions.

Corollary If $\det A \neq 0$ then $A^{-1} = (\det A)^{-1}\widetilde{A}^T$.