MATH 323

Linear Algebra

Lecture 21:
Orthogonal complement.
Orthogonal projection.
Least squares problems.

Orthogonality

Definition 1. Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\boxed{\mathbf{x} \cdot \mathbf{y} = 0.}$

Definition 2. A vector $\mathbf{x} \in \mathbb{R}^n$ is said to be **orthogonal** to a nonempty set $Y \subset \mathbb{R}^n$ (denoted $\mathbf{x} \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{y} \in Y$.

Definition 3. Nonempty sets $X, Y \subset \mathbb{R}^n$ are said to be **orthogonal** (denoted $X \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.

Orthogonal complement

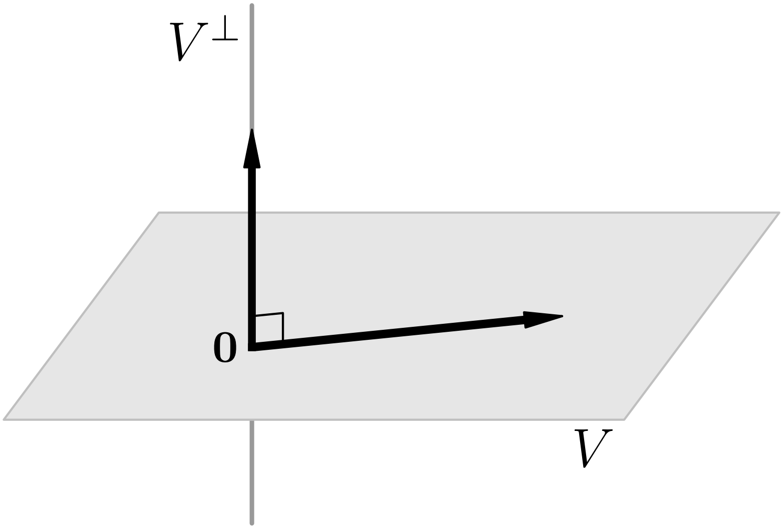
Definition. Let $S \subset \mathbb{R}^n$. The **orthogonal complement** of S, denoted $S^\perp$, is the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ that are orthogonal to S. That is, $S^\perp$ is the largest subset of $\mathbb{R}^n$ orthogonal to S.

Theorem 1 $S^\perp$ is a subspace of $\mathbb{R}^n$.

Note that $S \subset (S^\perp)^\perp$, hence $\mathrm{Span}(S) \subset (S^\perp)^\perp$.

Theorem 2 $(S^\perp)^\perp = \mathrm{Span}(S)$. In particular, for any subspace V we have $(V^\perp)^\perp = V$.

Example. Consider a line $L = \{(x, 0, 0) \mid x \in \mathbb{R}\}$ and a plane $\Pi = \{(0, y, z) \mid y, z \in \mathbb{R}\}$ in $\mathbb{R}^3$. Then $L^\perp = \Pi$ and $\Pi^\perp = L$.


$V^\perp$
$\mathbf{0}$
V

Fundamental subspaces

Definition. Given an $m\times n$ matrix A, let

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\},$$

$$R(A) = \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

$R(A)$ is the range of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$, $L(\mathbf{x}) = A\mathbf{x}$. $N(A)$ is the kernel of L.

Also, $N(A)$ is the nullspace of the matrix A while $R(A)$ is the column space of A. The row space of A is $R(A^T)$.

The subspaces $N(A), R(A^T) \subset \mathbb{R}^n$ and $R(A), N(A^T) \subset \mathbb{R}^m$ are **fundamental subspaces** associated to the matrix A.

Theorem $N(A) = R(A^T)^\perp$, $N(A^T) = R(A)^\perp$.
That is, the nullspace of a matrix is the orthogonal complement of its row space.

Proof: The equality $A\mathbf{x} = \mathbf{0}$ means that the vector $\mathbf{x}$ is orthogonal to rows of the matrix A. Therefore $N(A) = S^\perp$, where S is the set of rows of A. It remains to note that $S^\perp = \mathrm{Span}(S)^\perp = R(A^T)^\perp$.

Corollary Let V be a subspace of $\mathbb{R}^n$. Then $\dim V + \dim V^\perp = n$.

Proof: Pick a basis $\mathbf{v}_1, \dots, \mathbf{v}_k$ for V. Let A be the $k \times n$ matrix whose rows are vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$. Then $V = R(A^T)$, hence $V^\perp = N(A)$. Consequently, $\dim V$ and $\dim V^\perp$ are rank and nullity of A. Therefore $\dim V + \dim V^\perp$ equals the number of columns of A, which is n.

Problem. Let V be the plane spanned by vectors $\mathbf{v}_1 = (1, 1, 0)$ and $\mathbf{v}_2 = (0, 1, 1)$. Find $V^\perp$.

The orthogonal complement to V is the same as the orthogonal complement of the set $\{\mathbf{v}_1, \mathbf{v}_2\}$. A vector $\mathbf{u} = (x, y, z)$ belongs to the latter if and only if

$$\begin{cases} \mathbf{u} \cdot \mathbf{v}_1 = 0 \\ \mathbf{u} \cdot \mathbf{v}_2 = 0 \end{cases} \iff \begin{cases} x + y = 0 \\ y + z = 0 \end{cases}$$

Alternatively, the subspace V is the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

hence $V^\perp$ is the nullspace of A.

The general solution of the system (or, equivalently, the general element of the nullspace of A) is $(t, -t, t)$ $= t(1, -1, 1)$, $t \in \mathbb{R}$. Thus $V^\perp$ is the straight line spanned by the vector $(1, -1, 1)$.

Orthogonal projection

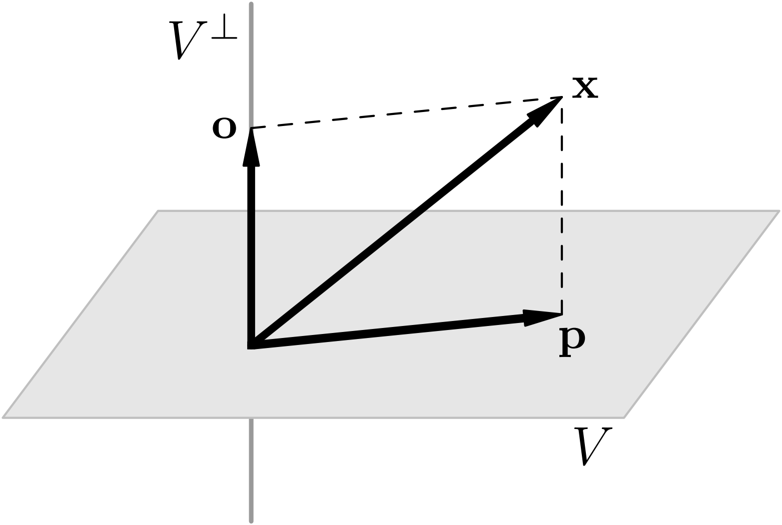
Theorem 1 Let V be a subspace of $\mathbb{R}^n$. Then any vector $\mathbf{x} \in \mathbb{R}^n$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o} \in V^\perp$.

Idea of the proof: Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a basis for V and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be a basis for $V^\perp$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_m$ is a linearly independent set. Hence it is a basis for $\mathbb{R}^n$.

In the above expansion, $\mathbf{p}$ is called the **orthogonal projection** of the vector $\mathbf{x}$ onto the subspace V.

Theorem 2 $\|\mathbf{x} - \mathbf{v}\| > \|\mathbf{x} - \mathbf{p}\|$ for any $\mathbf{v} \neq \mathbf{p}$ in V.

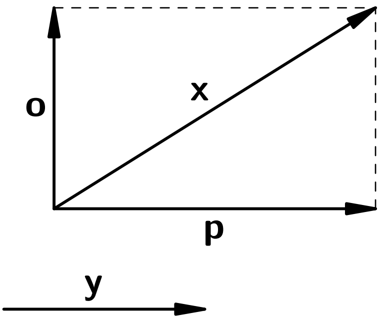
Thus $\|\mathbf{o}\| = \|\mathbf{x} - \mathbf{p}\| = \min_{\mathbf{v} \in V} \|\mathbf{x} - \mathbf{v}\|$ is the **distance** from the vector $\mathbf{x}$ to the subspace V.



$V^\perp$
x
o
p
V

Orthogonal projection onto a vector

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with $\mathbf{y} \neq \mathbf{0}$.
Then there exists a unique decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$ such that $\mathbf{p}$ is parallel to $\mathbf{y}$ and $\mathbf{o}$ is orthogonal to $\mathbf{y}$.



$\mathbf{p}$ = orthogonal projection of $\mathbf{x}$ onto $\mathbf{y}$

Orthogonal projection onto a vector

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with $\mathbf{y} \neq \mathbf{0}$.
Then there exists a unique decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$ such that $\mathbf{p}$ is parallel to $\mathbf{y}$ and $\mathbf{o}$ is orthogonal to $\mathbf{y}$.

We have $\mathbf{p} = \alpha \mathbf{y}$ for some $\alpha \in \mathbb{R}$. Then

$$0 = \mathbf{o} \cdot \mathbf{y} = (\mathbf{x} - \alpha \mathbf{y}) \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y} - \alpha \mathbf{y} \cdot \mathbf{y}.$$

$$\implies \quad \alpha = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \quad \implies \quad \boxed{\mathbf{p} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y}}$$

Problem. Find the distance from the point $\mathbf{x} = (3, 1)$ to the line spanned by $\mathbf{y} = (2, -1)$.

Consider the decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p}$ is parallel to $\mathbf{y}$ while $\mathbf{o} \perp \mathbf{y}$. The required distance is the length of the orthogonal component $\mathbf{o}$.

$$\mathbf{p} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y} = \frac{5}{5}(2, -1) = (2, -1),$$

$$\mathbf{o} = \mathbf{x} - \mathbf{p} = (3, 1) - (2, -1) = (1, 2), \quad \|\mathbf{o}\| = \sqrt{5}.$$

Problem. Find the point on the line $y = -x$ that is closest to the point $(3, 4)$.

The required point is the projection $\mathbf{p}$ of $\mathbf{v} = (3, 4)$ on the vector $\mathbf{w} = (1, -1)$ spanning the line $y = -x$.

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w} = \frac{-1}{2}(1, -1) = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Problem. Let Π be the plane spanned by vectors $\mathbf{v}_1 = (1,1,0)$ and $\mathbf{v}_2 = (0,1,1)$.

(i) Find the orthogonal projection of the vector $\mathbf{x} = (4,0,-1)$ onto the plane Π.

(ii) Find the distance from $\mathbf{x}$ to Π.

We have $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in \Pi$ and $\mathbf{o} \perp \Pi$. Then the orthogonal projection of $\mathbf{x}$ onto Π is $\mathbf{p}$ and the distance from $\mathbf{x}$ to Π is $\|\mathbf{o}\|$.

We have $\mathbf{p} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$ for some $\alpha, \beta \in \mathbb{R}$. Then $\mathbf{o} = \mathbf{x} - \mathbf{p} = \mathbf{x} - \alpha\mathbf{v}_1 - \beta\mathbf{v}_2$.

$$\begin{cases} \mathbf{o}\cdot\mathbf{v}_1 = 0 \\ \mathbf{o}\cdot\mathbf{v}_2 = 0 \end{cases} \iff \begin{cases} \alpha(\mathbf{v}_1\cdot\mathbf{v}_1) + \beta(\mathbf{v}_2\cdot\mathbf{v}_1) = \mathbf{x}\cdot\mathbf{v}_1 \\ \alpha(\mathbf{v}_1\cdot\mathbf{v}_2) + \beta(\mathbf{v}_2\cdot\mathbf{v}_2) = \mathbf{x}\cdot\mathbf{v}_2 \end{cases}$$

$$\mathbf{x} = (4, 0, -1), \;\; \mathbf{v}_1 = (1, 1, 0), \;\; \mathbf{v}_2 = (0, 1, 1)$$

$$\begin{cases} \alpha(\mathbf{v}_1 \cdot \mathbf{v}_1) + \beta(\mathbf{v}_2 \cdot \mathbf{v}_1) = \mathbf{x} \cdot \mathbf{v}_1 \\ \alpha(\mathbf{v}_1 \cdot \mathbf{v}_2) + \beta(\mathbf{v}_2 \cdot \mathbf{v}_2) = \mathbf{x} \cdot \mathbf{v}_2 \end{cases}$$

$$\iff \begin{cases} 2\alpha + \beta = 4 \\ \alpha + 2\beta = -1 \end{cases} \iff \begin{cases} \alpha = 3 \\ \beta = -2 \end{cases}$$

$$\mathbf{p} = 3\mathbf{v}_1 - 2\mathbf{v}_2 = (3, 1, -2)$$

$$\mathbf{o} = \mathbf{x} - \mathbf{p} = (1, -1, 1)$$

$$\|\mathbf{o}\| = \sqrt{3}$$

Overdetermined system of linear equations:

$$\begin{cases} x + 2y = 3 \\ 3x + 2y = 5 \\ x + y = 2.09 \end{cases} \iff \begin{cases} x + 2y = 3 \\ -4y = -4 \\ -y = -0.91 \end{cases}$$

No solution: inconsistent system

Assume that a solution (x_0, y_0) does exist but the system is not quite accurate, namely, there may be some errors in the right-hand sides.

Problem. Find a good approximation of (x_0, y_0).

One approach is the **least squares fit**. Namely, we look for a pair (x, y) that minimizes the sum $(x + 2y - 3)^2 + (3x + 2y - 5)^2 + (x + y - 2.09)^2$.

Least squares solution

System of linear equations:

$$\left\{ \begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right. \iff A\mathbf{x} = \mathbf{b}$$

For any $\mathbf{x} \in \mathbb{R}^n$ define a **residual** $r(\mathbf{x}) = \mathbf{b} - A\mathbf{x}$.

The **least squares solution x** to the system is the one that minimizes $\|r(\mathbf{x})\|$ (or, equivalently, $\|r(\mathbf{x})\|^2$).

$$\|r(\mathbf{x})\|^2 = \sum_{i=1}^{m} (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i)^2$$

Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$.

Theorem A vector $\hat{\mathbf{x}}$ is a least squares solution of the system $A\mathbf{x} = \mathbf{b}$ if and only if it is a solution of the associated **normal system** $\boxed{A^T A\mathbf{x} = A^T\mathbf{b}.}$

Proof: $A\mathbf{x}$ is an arbitrary vector in $R(A)$, the column space of A. Hence the length of $r(\mathbf{x}) = \mathbf{b} - A\mathbf{x}$ is minimal if $A\mathbf{x}$ is the orthogonal projection of $\mathbf{b}$ onto $R(A)$. That is, if $r(\mathbf{x})$ is orthogonal to $R(A)$.

We know that $\{\text{row space}\}^\perp = \{\text{nullspace}\}$ for any matrix. In particular, $R(A)^\perp = N(A^T)$, the nullspace of the transpose matrix of A. Thus $\hat{\mathbf{x}}$ is a least squares solution if and only if

$$A^T r(\hat{\mathbf{x}}) = \mathbf{0} \iff A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0} \iff A^T A\hat{\mathbf{x}} = A^T\mathbf{b}.$$

Corollary The normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent.

Problem. Find the least squares solution to

$$\begin{cases} x + 2y = 3 \\ 3x + 2y = 5 \\ x + y = 2.09 \end{cases}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 2.09 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 2.09 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 9 \\ 9 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20.09 \\ 18.09 \end{pmatrix} \iff \begin{cases} x = 1 \\ y = 1.01 \end{cases}$$

Problem. Find the constant function that is the least squares fit to the following data

x	0	1	2	3
$f(x)$	1	0	1	2

$$f(x) = c \implies \begin{cases} c = 1 \\ c = 0 \\ c = 1 \\ c = 2 \end{cases} \implies \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (c) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$(1,1,1,1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (c) = (1,1,1,1) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$c = \frac{1}{4}(1+0+1+2) = 1$ (mean arithmetic value)

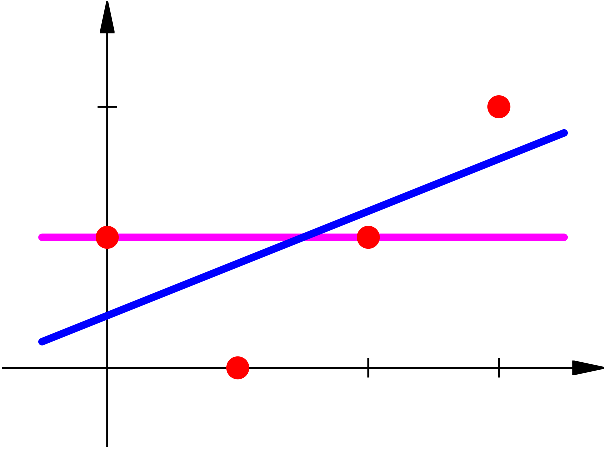
Problem. Find the linear polynomial that is the least squares fit to the following data

x	0	1	2	3
$f(x)$	1	0	1	2

$$f(x) = c_1 + c_2 x \implies \begin{cases} c_1 = 1 \\ c_1 + c_2 = 0 \\ c_1 + 2c_2 = 1 \\ c_1 + 3c_2 = 2 \end{cases} \implies \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \iff \begin{cases} c_1 = 0.4 \\ c_2 = 0.4 \end{cases}$$



Problem. Find the quadratic polynomial that is the least squares fit to the following data

x	0	1	2	3
$f(x)$	1	0	1	2

$$f(x) = c_1 + c_2 x + c_3 x^2$$

$$\Longrightarrow \begin{cases} c_1 = 1 \\ c_1 + c_2 + c_3 = 0 \\ c_1 + 2c_2 + 4c_3 = 1 \\ c_1 + 3c_2 + 9c_3 = 2 \end{cases} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 22 \end{pmatrix} \Longleftrightarrow \begin{cases} c_1 = 0.9 \\ c_2 = -1.1 \\ c_3 = 0.5 \end{cases}$$