MATH 323 Linear Algebra

Lecture 5: Inverse matrix.

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The $n \times n$ identity matrix is denoted I_n or simply I.

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In general, $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$

Theorem. Let A be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. We can **add**, **subtract**, and **multiply** elements of $\mathcal{M}_n(\mathbb{R})$. What about **division**?

Definition. Let $A \in \mathcal{M}_n(\mathbb{R})$. Suppose there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

Then the matrix A is called **invertible** and B is called the **inverse** of A (denoted A^{-1}).

A non-invertible square matrix is called **singular**.

$$AA^{-1} = A^{-1}A = I$$

Examples

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C^{2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Thus $A^{-1} = B, \quad B^{-1} = A, \text{ and } C^{-1} = C.$

Basic properties of inverse matrices

• If $B = A^{-1}$ then $A = B^{-1}$. In other words, if A is invertible, so is A^{-1} , and $A = (A^{-1})^{-1}$.

• The inverse matrix (if it exists) is unique. Moreover, if AB = CA = I for some $n \times n$ matrices B and C, then $B = C = A^{-1}$.

Indeed, B = IB = (CA)B = C(AB) = CI = C.

• If $n \times n$ matrices A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

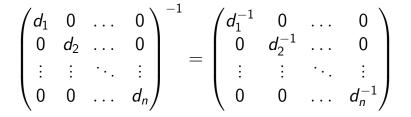
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$

 $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$

• Similarly, $(A_1A_2...A_k)^{-1} = A_k^{-1}...A_2^{-1}A_1^{-1}.$

Inverting diagonal matrices

Theorem A diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$. If D is invertible then $D^{-1} = \text{diag}(d_1^{-1}, \ldots, d_n^{-1})$.



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If D is invertible then $D^{-1} = \operatorname{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

Proof: If all $d_i \neq 0$ then, clearly, $\operatorname{diag}(d_1, \ldots, d_n) \operatorname{diag}(d_1^{-1}, \ldots, d_n^{-1}) = \operatorname{diag}(1, \ldots, 1) = I$, $\operatorname{diag}(d_1^{-1}, \ldots, d_n^{-1}) \operatorname{diag}(d_1, \ldots, d_n) = \operatorname{diag}(1, \ldots, 1) = I$. Now suppose that $d_i = 0$ for some *i*. Then for any $n \times n$ matrix *B* the *i*th row of the matrix *DB* is a zero row. Hence $DB \neq I$ as *I* has no zero rows.

Inverting 2×2 matrices

Definition. The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is det A = ad - bc.

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if det $A \neq 0$.

If det $A \neq 0$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$ **Theorem** A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if det $A \neq 0$. If det $A \neq 0$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$ *Proof:* Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then $AB = BA = \begin{pmatrix} ad-bc & 0\\ 0 & ad-bc \end{pmatrix} = (ad-bc)I_2.$

In the case det $A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$. In the case det A = 0, the matrix A is not invertible as otherwise $AB = O \implies A^{-1}(AB) = A^{-1}O = O$ $\implies (A^{-1}A)B = O \implies I_2B = O \implies B = O$ $\implies A = O$, but the zero matrix is singular.

Problem. Solve a system
$$\begin{cases} 4x + 3y = 5, \\ 3x + 2y = -1. \end{cases}$$

This system is equivalent to a matrix equation $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

We have det $A = -1 \neq 0$. Hence A is invertible.

$$A\mathbf{x} = \mathbf{b} \implies A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \implies (A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$$
$$\implies \mathbf{x} = A^{-1}\mathbf{b}.$$

Conversely, $\mathbf{x} = A^{-1}\mathbf{b} \implies A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 19 \end{pmatrix}$$

System of *n* linear equations in *n* variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \dots \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \iff A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Theorem If the matrix A is invertible then the system has a unique solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.

General results on inverse matrices

Theorem 1 Given an $n \times n$ matrix A, the following conditions are equivalent:

(i) A is invertible;

(ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$; (iii) the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any *n*-dimensional column vector **b**;

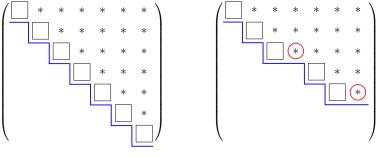
(iv) the row echelon form of A has no zero rows;

(v) the reduced row echelon form of A is the identity matrix.

Theorem 2 Suppose that a sequence of elementary row operations converts a matrix *A* into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1} .

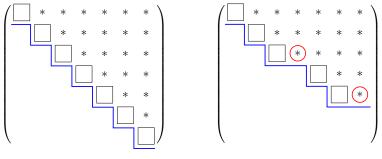
Row echelon form of a square matrix:



invertible case

noninvertible case

For any matrix in row echelon form, the number of columns with leading entries equals the number of rows with leading entries. For a square matrix, also the number of columns *without* leading entries (i.e., the number of free variables in a related system of linear equations) equals the number of rows *without* leading entries (i.e., zero rows). Row echelon form of a square matrix:



invertible case

noninvertible case

Hence the row echelon form of a square matrix A is either strict triangular or else it has a zero row. In the former case, the equation $A\mathbf{x} = \mathbf{b}$ always has a unique solution. In the latter case, $A\mathbf{x} = \mathbf{b}$ never has a unique solution. Also, in the former case the reduced row echelon form of A is I.

Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
.

To check whether A is invertible, we convert it to row echelon form.

Interchange the 1st row with the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & -2 & 0 \\ -2 & 3 & 0 \end{pmatrix}$$

Add -3 times the 1st row to the 2nd row:

$$egin{pmatrix} 1 & 0 & 1 \ 0 & -2 & -3 \ -2 & 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Add 2 times the 1st row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 3 & 2 \end{pmatrix}$$

Multiply the 2nd row by -0.5:

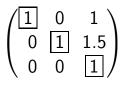
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

Add -3 times the 2nd row to the 3rd row: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & -2.5 \end{pmatrix}$

Multiply the 3rd row by -0.4:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{pmatrix}$$



We already know that the matrix A is invertible. Let's proceed towards reduced row echelon form.

Add -1.5 times the 3rd row to the 2nd row: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Add -1 times the 3rd row to the 1st row: $\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$ To obtain A^{-1} , we need to apply the following sequence of elementary row operations to the identity matrix:

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by -0.5,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by -0.4,
- add -1.5 times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

A convenient way to compute the inverse matrix A^{-1} is to merge the matrices A and I into one 3×6 matrix $(A \mid I)$, and apply elementary row operations to this new matrix.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$(A \mid I) = \begin{pmatrix} 3 & -2 & 0 \mid 1 & 0 & 0 \\ 1 & 0 & 1 \mid 0 & 1 & 0 \\ -2 & 3 & 0 \mid 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 & 0 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ -2 & 3 & 0 & | & 0 & 0 & 1 \end{pmatrix}$$

Interchange the 1st row with the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Add -3 times the 1st row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Add 2 times the 1st row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{pmatrix}$$

Multiply the 2nd row by -0.5:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \\ \end{bmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -0.5 & 1.5 & 0 \\ 0 & 2 & 1 \\ \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{pmatrix}$$

Add -3 times the 2nd row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & -2.5 & 1.5 & -2.5 & 1 \end{pmatrix}$$

Multiply the 3rd row by -0.4:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{pmatrix}$$

Add -1.5 times the 3rd row to the 2nd row: $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{pmatrix}$

Add -1 times the 3rd row to the 1st row: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.6 & 1 & -0.4 \end{pmatrix} = (I | A^{-1})$

Thus
$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$