

MATH 323

Linear Algebra

**Lecture 12:**

**Basis and dimension (continued).**

**Rank of a matrix.**

## Basis

*Definition.* Let  $V$  be a vector space. Any linearly independent spanning set for  $V$  is called a **basis**.

**Theorem** A nonempty set  $S \subset V$  is a basis for  $V$  if and only if any vector  $\mathbf{v} \in V$  is *uniquely represented* as a linear combination

$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$ , where  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are distinct vectors from  $S$  and  $r_1, \dots, r_k \in \mathbb{R}$ .

*Remark on uniqueness.* Expansions  $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2$ ,  $\mathbf{v} = -\mathbf{v}_2 + 2\mathbf{v}_1$ , and  $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2 + 0\mathbf{v}_3$  are considered the same.

## Dimension

**Theorem 1** Any vector space has a basis.

**Theorem 2** If a vector space  $V$  has a finite basis, then all bases for  $V$  are finite and have the same number of elements.

*Definition.* The **dimension** of a vector space  $V$ , denoted  $\dim V$ , is the number of elements in any of its bases.

*Examples.* •  $\dim \mathbb{R}^n = n$

- $\mathcal{M}_{m,n}(\mathbb{R})$ : the space of  $m \times n$  matrices;  $\dim \mathcal{M}_{m,n} = mn$
- $\mathcal{P}_n$ : polynomials of degree less than  $n$ ;  $\dim \mathcal{P}_n = n$
- $\mathcal{P}$ : the space of all polynomials;  $\dim \mathcal{P} = \infty$
- $\{\mathbf{0}\}$ : the trivial vector space;  $\dim \{\mathbf{0}\} = 0$

**Problem.** Find the dimension of the plane  $x + 2z = 0$  in  $\mathbb{R}^3$ .

The general solution of the equation  $x + 2z = 0$  is

$$\begin{cases} x = -2s \\ y = t \\ z = s \end{cases} \quad (t, s \in \mathbb{R})$$

That is,  $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$ .

Hence the plane is the span of vectors  $\mathbf{v}_1 = (0, 1, 0)$  and  $\mathbf{v}_2 = (-2, 0, 1)$ . These vectors are linearly independent as they are not parallel.

Thus  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a basis so that the dimension of the plane is 2.

## How to find a basis?

**Theorem** Let  $S$  be a subset of a vector space  $V$ . Then the following conditions are equivalent:

- (i)  $S$  is a linearly independent spanning set for  $V$ , i.e., a basis;
- (ii)  $S$  is a minimal spanning set for  $V$ ;
- (iii)  $S$  is a maximal linearly independent subset of  $V$ .

“Minimal spanning set” means “remove any element from this set, and it is no longer a spanning set”.

“Maximal linearly independent subset” means “add any element of  $V$  to this set, and it will become linearly dependent”.

**Theorem** Let  $V$  be a vector space. Then

- (i) any spanning set for  $V$  can be reduced to a minimal spanning set;
- (ii) any linearly independent subset of  $V$  can be extended to a maximal linearly independent set.

**Corollary 1** Any spanning set contains a basis while any linearly independent set is contained in a basis.

**Corollary 2** A vector space is finite-dimensional if and only if it is spanned by a finite set.

## How to find a basis?

*Approach 1.* Get a spanning set for the vector space, then reduce this set to a basis dropping one vector at a time.

**Proposition** Let  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$  be a spanning set for a vector space  $V$ . If  $\mathbf{v}_0$  is a linear combination of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  then  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is also a spanning set for  $V$ .

Indeed, if  $\mathbf{v}_0 = r_1\mathbf{v}_1 + \dots + r_k\mathbf{v}_k$ , then

$$\begin{aligned} t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \dots + t_k\mathbf{v}_k &= \\ &= (t_0r_1 + t_1)\mathbf{v}_1 + \dots + (t_0r_k + t_k)\mathbf{v}_k. \end{aligned}$$

## How to find a basis?

*Approach 2.* Build a maximal linearly independent set adding one vector at a time.

If the vector space  $V$  is trivial, it has the empty basis. If  $V \neq \{\mathbf{0}\}$ , pick any vector  $\mathbf{v}_1 \neq \mathbf{0}$ . If  $\mathbf{v}_1$  spans  $V$ , it is a basis. Otherwise pick any vector  $\mathbf{v}_2 \in V$  that is not in the span of  $\mathbf{v}_1$ . If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  span  $V$ , they constitute a basis. Otherwise pick any vector  $\mathbf{v}_3 \in V$  that is not in the span of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . And so on...

*Modifications.* Instead of the empty set, we can start with any linearly independent set (if we are given one). If we are given a spanning set  $S$ , it is enough to pick new vectors only in  $S$ .

*Remark.* This inductive procedure works for finite-dimensional vector spaces. There is an analogous procedure for infinite-dimensional spaces (*transfinite induction*).



Vectors  $\mathbf{v}_1 = (0, 1, 0)$  and  $\mathbf{v}_2 = (-2, 0, 1)$  are linearly independent.

**Problem.** Extend the set  $\{\mathbf{v}_1, \mathbf{v}_2\}$  to a basis for  $\mathbb{R}^3$ .

Our task is to find a vector  $\mathbf{v}_3$  that is not a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

Then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  will be a basis for  $\mathbb{R}^3$ .

*Hint 1.*  $\mathbf{v}_1$  and  $\mathbf{v}_2$  span the plane  $x + 2z = 0$ .

The vector  $\mathbf{v}_3 = (1, 1, 1)$  does not lie in the plane  $x + 2z = 0$ , hence it is not a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . Thus  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis for  $\mathbb{R}^3$ .

Vectors  $\mathbf{v}_1 = (0, 1, 0)$  and  $\mathbf{v}_2 = (-2, 0, 1)$  are linearly independent.

**Problem.** Extend the set  $\{\mathbf{v}_1, \mathbf{v}_2\}$  to a basis for  $\mathbb{R}^3$ .

Our task is to find a vector  $\mathbf{v}_3$  that is not a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . Then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  will be a basis for  $\mathbb{R}^3$ .

*Hint 2.* Since vectors  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ , and  $\mathbf{e}_3 = (0, 0, 1)$  form a spanning set for  $\mathbb{R}^3$ , at least one of them can be chosen as  $\mathbf{v}_3$ .

Let us check that  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$  and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$  are two bases for  $\mathbb{R}^3$ :

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \quad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$

**Problem.** Find a basis for the vector space  $V$  spanned by vectors  $\mathbf{w}_1 = (1, 1, 0)$ ,  $\mathbf{w}_2 = (0, 1, 1)$ ,  $\mathbf{w}_3 = (2, 3, 1)$ , and  $\mathbf{w}_4 = (1, 1, 1)$ .

To pare this spanning set, we need to find a relation of the form  $r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + r_3\mathbf{w}_3 + r_4\mathbf{w}_4 = \mathbf{0}$ , where  $r_i \in \mathbb{R}$  are not all equal to zero. Equivalently,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system of linear equations for  $r_1, r_2, r_3, r_4$ , we apply row reduction.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 \rightarrow \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{reduced row echelon form})$$

$$\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \iff \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}$$

General solution:  $(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$ ,  $t \in \mathbb{R}$ .

Particular solution:  $(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$ .

**Problem.** Find a basis for the vector space  $V$  spanned by vectors  $\mathbf{w}_1 = (1, 1, 0)$ ,  $\mathbf{w}_2 = (0, 1, 1)$ ,  $\mathbf{w}_3 = (2, 3, 1)$ , and  $\mathbf{w}_4 = (1, 1, 1)$ .

We have obtained that  $2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$ .

Hence any of vectors  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$  can be dropped.

For instance,  $V = \text{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4)$ .

Let us check whether vectors  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4$  are linearly independent:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0.$$

They are!!! It follows that  $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$  is a basis for  $V$ . Also, it follows that  $V = \mathbb{R}^3$ .

## Row space of a matrix

*Definition.* The **row space** of an  $m \times n$  matrix  $A$  is the subspace of  $\mathbb{R}^n$  spanned by rows of  $A$ .

The dimension of the row space is called the **rank** of the matrix  $A$ .

**Theorem 1** The rank of a matrix  $A$  is the maximal number of linearly independent rows in  $A$ .

**Theorem 2** Elementary row operations do not change the row space of a matrix.

**Theorem 3** If a matrix  $A$  is in row echelon form, then the nonzero rows of  $A$  are linearly independent.

**Corollary** The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

**Theorem** Elementary row operations do not change the row space of a matrix.

*Proof:* Suppose that  $A$  and  $B$  are  $m \times n$  matrices such that  $B$  is obtained from  $A$  by an elementary row operation. Let  $\mathbf{a}_1, \dots, \mathbf{a}_m$  be the rows of  $A$  and  $\mathbf{b}_1, \dots, \mathbf{b}_m$  be the rows of  $B$ . We have to show that  $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_m)$ .

Observe that any row  $\mathbf{b}_i$  of  $B$  belongs to  $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$ . Indeed, either  $\mathbf{b}_i = \mathbf{a}_j$  for some  $1 \leq j \leq m$ , or  $\mathbf{b}_i = r\mathbf{a}_i$  for some scalar  $r \neq 0$ , or  $\mathbf{b}_i = \mathbf{a}_i + r\mathbf{a}_j$  for some  $j \neq i$  and  $r \in \mathbb{R}$ .

It follows that  $\text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_m) \subset \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$ .

Now the matrix  $A$  can also be obtained from  $B$  by an elementary row operation. By the above,

$$\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \subset \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_m).$$

**Problem.** Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Elementary row operations do not change the row space. Let us convert  $A$  to row echelon form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Vectors  $(1, 1, 0)$ ,  $(0, 1, 1)$ , and  $(0, 0, 1)$  form a basis for the row space of  $A$ . Thus the rank of  $A$  is 3.

It follows that the row space of  $A$  is the entire space  $\mathbb{R}^3$ .

**Problem.** Find a basis for the vector space  $V$  spanned by vectors  $\mathbf{w}_1 = (1, 1, 0)$ ,  $\mathbf{w}_2 = (0, 1, 1)$ ,  $\mathbf{w}_3 = (2, 3, 1)$ , and  $\mathbf{w}_4 = (1, 1, 1)$ .

The vector space  $V$  is the row space of a matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

According to the solution of the previous problem, vectors  $(1, 1, 0)$ ,  $(0, 1, 1)$ , and  $(0, 0, 1)$  form a basis for  $V$ .

## Column space of a matrix

*Definition.* The **column space** of an  $m \times n$  matrix  $A$  is the subspace of  $\mathbb{R}^m$  spanned by columns of  $A$ .

**Theorem 1** The column space of a matrix  $A$  coincides with the row space of the transpose matrix  $A^T$ .

**Theorem 2** Elementary row operations do not change linear relations between columns of a matrix.

**Theorem 3** Elementary row operations do not change the dimension of the column space of a matrix (however they can change the column space).

**Theorem 4** If a matrix is in row echelon form, then the columns with leading entries form a basis for the column space.

**Corollary** For any matrix, the row space and the column space have the same dimension.