

MATH 323

Linear Algebra

Lecture 15:

Change of basis.

Linear transformations.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients x_1, x_2, \dots, x_n are called the **coordinates** of \mathbf{v} with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

The mapping

$$\text{vector } \mathbf{v} \mapsto \text{its coordinates } (x_1, x_2, \dots, x_n)$$

is a one-to-one correspondence between V and \mathbb{R}^n . This correspondence respects linear operations in V and in \mathbb{R}^n .

Change of coordinates

Given a vector $\mathbf{v} \in \mathbb{R}^2$, let (x, y) be its standard coordinates, i.e., coordinates with respect to the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and let (x', y') be its coordinates with respect to the basis $\mathbf{u}_1 = (3, 1)$, $\mathbf{u}_2 = (2, 1)$.

Problem. Find a relation between (x, y) and (x', y') .

By definition, $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{u}_1 + y'\mathbf{u}_2$.

In standard coordinates,

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= x' \begin{pmatrix} 3 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ \implies \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Change of coordinates in \mathbb{R}^n

The usual (standard) coordinates of a vector $\mathbf{v} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ are coordinates relative to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for \mathbb{R}^n and $(x'_1, x'_2, \dots, x'_n)$ be the coordinates of the same vector \mathbf{v} with respect to this basis. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix},$$

where the matrix $U = (u_{ij})$ does not depend on the vector \mathbf{v} . Namely, columns of U are coordinates of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ with respect to the standard basis. U is called the **transition matrix** from the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. The inverse matrix U^{-1} is called the **transition matrix** from $\mathbf{e}_1, \dots, \mathbf{e}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$.

Problem. Find coordinates of the vector $\mathbf{x} = (1, 2, 3)$ with respect to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

The nonstandard coordinates (x', y', z') of \mathbf{x} satisfy

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = U \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where U is the transition matrix from the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

The transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$U_0 = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \left(\begin{array}{c|c|c} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right).$$

The transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the inverse matrix: $U = U_0^{-1}$.

The inverse matrix can be computed using row reduction.

$$(U_0 | I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array} \right) = (I | U_0^{-1})$$

Thus

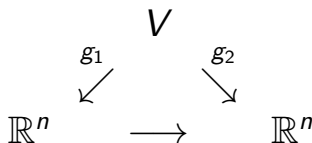
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Change of coordinates: general case

Let V be a vector space of dimension n .

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.



The composition $g_2 \circ g_1^{-1}$ is a transformation of \mathbb{R}^n .

It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

Problem. Find the transition matrix from the basis $p_1(x) = 1$, $p_2(x) = x + 1$, $p_3(x) = (x + 1)^2$ to the basis $q_1(x) = 1$, $q_2(x) = x$, $q_3(x) = x^2$ for the vector space \mathcal{P}_3 .

We have to find coordinates of the polynomials p_1, p_2, p_3 with respect to the basis q_1, q_2, q_3 :

$$p_1(x) = 1 = q_1(x),$$

$$p_2(x) = x + 1 = q_1(x) + q_2(x),$$

$$p_3(x) = (x+1)^2 = x^2 + 2x + 1 = q_1(x) + 2q_2(x) + q_3(x).$$

Hence the transition matrix is
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the polynomial identity

$$a_1 + a_2(x + 1) + a_3(x + 1)^2 = b_1 + b_2x + b_3x^2$$

is equivalent to the relation

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Problem. Find the transition matrix from the basis $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, 2, 1)$ to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

It is convenient to make a two-step transition: first from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and then from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Let U_1 be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and U_2 be the transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \implies$ coordinates \mathbf{x}

Basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \implies$ coordinates $U_1\mathbf{x}$

Basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \implies$ coordinates $U_2^{-1}(U_1\mathbf{x}) = (U_2^{-1}U_1)\mathbf{x}$

Thus the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is $U_2^{-1}U_1$.

$$\begin{aligned} U_2^{-1}U_1 &= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}. \end{aligned}$$

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2 , a mapping $L : V_1 \rightarrow V_2$ is **linear** if

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = rL(\mathbf{x})$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

A linear mapping $\ell : V \rightarrow \mathbb{R}$ is called a **linear functional** on V .

If $V_1 = V_2$ (or if both V_1 and V_2 are functional spaces) then a linear mapping $L : V_1 \rightarrow V_2$ is called a **linear operator**.

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2 , a mapping $L : V_1 \rightarrow V_2$ is **linear** if

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = rL(\mathbf{x})$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Remark. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = ax + b$ is a linear transformation of the vector space \mathbb{R} if and only if $b = 0$.

Basic properties of linear transformations

Let $L : V_1 \rightarrow V_2$ be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$
for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.

$$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2),$$

$$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) = \\ = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2) + r_3L(\mathbf{v}_3), \text{ and so on.}$$

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2 , respectively.

$$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2.$$

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

$$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v}).$$

Examples of linear mappings

- *Scaling* $L : V \rightarrow V$, $L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{R}$.

$$L(\mathbf{x} + \mathbf{y}) = s(\mathbf{x} + \mathbf{y}) = s\mathbf{x} + s\mathbf{y} = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = s(r\mathbf{x}) = r(s\mathbf{x}) = rL(\mathbf{x}).$$

- *Dot product with a fixed vector*

$$\ell : \mathbb{R}^n \rightarrow \mathbb{R}, \ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0, \text{ where } \mathbf{v}_0 \in \mathbb{R}^n.$$

$$\ell(\mathbf{x} + \mathbf{y}) = (\mathbf{x} + \mathbf{y}) \cdot \mathbf{v}_0 = \mathbf{x} \cdot \mathbf{v}_0 + \mathbf{y} \cdot \mathbf{v}_0 = \ell(\mathbf{x}) + \ell(\mathbf{y}),$$

$$\ell(r\mathbf{x}) = (r\mathbf{x}) \cdot \mathbf{v}_0 = r(\mathbf{x} \cdot \mathbf{v}_0) = r\ell(\mathbf{x}).$$

- *Cross product with a fixed vector*

$$L : \mathbb{R}^3 \rightarrow \mathbb{R}^3, L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0, \text{ where } \mathbf{v}_0 \in \mathbb{R}^3.$$

- *Multiplication by a fixed matrix*

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^m, L(\mathbf{v}) = A\mathbf{v}, \text{ where } A \text{ is an } m \times n \text{ matrix and all vectors are column vectors.}$$

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$$\ell : F(\mathbb{R}) \rightarrow \mathbb{R}, \quad \ell(f) = f(a), \quad \text{where } a \in \mathbb{R}.$$

- *Multiplication by a fixed function*

$$L : F(\mathbb{R}) \rightarrow F(\mathbb{R}), \quad L(f) = gf, \quad \text{where } g \in F(\mathbb{R}).$$

- *Differentiation* $D : C^1(\mathbb{R}) \rightarrow C(\mathbb{R}), \quad D(f) = f'.$

$$D(f + g) = (f + g)' = f' + g' = D(f) + D(g),$$

$$D(rf) = (rf)' = rf' = rD(f).$$

- *Integration over a finite interval*

$$\ell : C(\mathbb{R}) \rightarrow \mathbb{R}, \quad \ell(f) = \int_a^b f(x) dx, \quad \text{where}$$

$$a, b \in \mathbb{R}, \quad a < b.$$

More properties of linear mappings

- If a linear mapping $L : V \rightarrow W$ is invertible then the inverse mapping $L^{-1} : W \rightarrow V$ is also linear.
- If $L : V \rightarrow W$ and $M : W \rightarrow X$ are linear mappings then the composition $M \circ L : V \rightarrow X$ is also linear.
- If $L_1 : V \rightarrow W$ and $L_2 : V \rightarrow W$ are linear mappings then the sum $L_1 + L_2$ is also linear.

Linear differential operators

- *Ordinary differential operator*

$$L : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on \mathbb{R} .

That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- *Laplace's operator* $\Delta : C^\infty(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Linear integral operators

- *Anti-derivative*

$$L : C[a, b] \rightarrow C^1[a, b], \quad (Lf)(x) = \int_a^x f(y) dy.$$

- *Hilbert-Schmidt operator*

$$L : C[a, b] \rightarrow C[c, d], \quad (Lf)(x) = \int_a^b K(x, y)f(y) dy,$$

where $K \in C([c, d] \times [a, b])$.

- *Laplace transform*

$$\mathcal{L} : BC(0, \infty) \rightarrow C(0, \infty), \quad (\mathcal{L}f)(x) = \int_0^{\infty} e^{-xy} f(y) dy.$$