MATH 323

Linear Algebra

Lecture 16:
Range and kernel.
General linear equations.
Multiplication by a matrix as a linear map.

Linear transformation

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear mappings:

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1 L(\mathbf{v}_1) + \cdots + r_k L(\mathbf{v}_k)$

for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

Examples of linear mappings

- *Scaling* $L : V \to V$, $L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{R}$.
- *Dot product with a fixed vector*
$\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^n$.
- *Cross product with a fixed vector*
$L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^3$.
- *Multiplication by a fixed matrix*
$L : \mathbb{R}^n \to \mathbb{R}^m$, $L(\mathbf{v}) = A\mathbf{v}$, where A is an $m\times n$ matrix and all vectors are column vectors.
- *Coordinate mapping*
$L : V \to \mathbb{R}^n$, $L(\mathbf{v}) =$ coordinates of $\mathbf{v}$ relative to an ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for the vector space V.

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$\ell : F(\mathbb{R}) \to \mathbb{R}, \;\; \ell(f) = f(a), \;$ where $\; a \in \mathbb{R}.$

- *Multiplication by a fixed function*

$L : F(\mathbb{R}) \to F(\mathbb{R}), \;\; L(f) = gf, \;$ where $\; g \in F(\mathbb{R}).$

- *Differentiation* $\; D : C^1(\mathbb{R}) \to C(\mathbb{R}), \;\; D(f) = f'.$

- *Integration over a finite interval*

$\ell : C(\mathbb{R}) \to \mathbb{R}, \;\; \ell(f) = \displaystyle\int_a^b f(x)\,dx, \;$ where $a, b \in \mathbb{R}, \;\; a < b.$

- *Change of the variable*

$L : F(\mathbb{R}) \to F(\mathbb{R}), \;\; L(f) = f \circ \phi, \;$ where $\; \phi \in F(\mathbb{R}).$

Examples. $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices.

- $\alpha : \mathcal{M}_{m,n}(\mathbb{R}) \to \mathcal{M}_{n,m}(\mathbb{R})$, $\alpha(A) = A^T$.

$\alpha(A + B) = \alpha(A) + \alpha(B) \iff (A + B)^T = A^T + B^T$.
$\alpha(rA) = r\,\alpha(A) \iff (rA)^T = rA^T$.
Hence α is linear.

- $\beta : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}$, $\beta(A) = \det A$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We have $\det(A) = \det(B) = 0$ while $\det(A + B) = 1$.
Hence $\beta(A + B) \neq \beta(A) + \beta(B)$ so that β is not linear.

Range and kernel

Let V, W be vector spaces and $L : V \to W$ be a linear map.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L, denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem (i) If V_0 is a subspace of V then $L(V_0)$ is a subspace of W. **(ii)** If W_0 is a subspace of W then $L^{-1}(W_0)$ is a subspace of V.

Corollary (i) The range of L is a subspace of W.
(ii) The kernel of L is a subspace of V.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The kernel $\ker(L)$ is the nullspace of the matrix.

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The range $L(\mathbb{R}^3)$ is the column space of the matrix.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The range of L is spanned by vectors $(1,1,1)$, $(0,2,0)$, and $(-1,-1,-1)$. It follows that $L(\mathbb{R}^3)$ is the plane spanned by $(1,1,1)$ and $(0,1,0)$.

To find $\ker(L)$, we apply row reduction to the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(x,y,z) \in \ker(L)$ if $x - z = y = 0$.
It follows that $\ker(L)$ is the line spanned by $(1,0,1)$.

More examples

- $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R}),\ \ L(A) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A.$

$$L \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}.$$

The range of L is the subspace of matrices with the zero second row, $\ker L$ is the same as the range $\implies L(L(A)) = O$.

- $D : \mathcal{P}_4 \to \mathcal{P}_4,\ \ (Dp)(x) = p'(x).$

$$p(x) = ax^3 + bx^2 + cx + d \implies (Dp)(x) = 3ax^2 + 2bx + c$$

The range of D is $\mathcal{P}_3$, $\ker D = \mathcal{P}_1$.

Example. $L: C^3(\mathbb{R}) \to C(\mathbb{R}),\ L(u) = u''' - 2u'' + u'$.

According to the theory of differential equations, the initial value problem

$$\begin{cases} u'''(x) - 2u''(x) + u'(x) = g(x), \quad x \in \mathbb{R}, \\ u(a) = b_0, \\ u'(a) = b_1, \\ u''(a) = b_2 \end{cases}$$

has a unique solution for any $g \in C(\mathbb{R})$ and any $b_0, b_1, b_2 \in \mathbb{R}$. It follows that $L(C^3(\mathbb{R})) = C(\mathbb{R})$.

Also, the initial data evaluation $I(u) = (u(a), u'(a), u''(a))$, which is a linear mapping $I : C^3(\mathbb{R}) \to \mathbb{R}^3$, becomes invertible when restricted to $\ker(L)$. Hence $\dim \ker(L) = 3$ since any invertible linear transformation maps a basis to a basis.

It is easy to check that $L(xe^x) = L(e^x) = L(1) = 0$. Besides, the functions xe^x, e^x, and 1 are linearly independent (use Wronskian). It follows that $\ker(L) = \mathrm{Span}(xe^x, e^x, 1)$.

General linear equation

Definition. A **linear equation** is an equation of the form

$$\boxed{L(\mathbf{x}) = \mathbf{b},}$$

where $L : V \to W$ is a linear mapping, $\mathbf{b}$ is a given vector from W, and $\mathbf{x}$ is an unknown vector from V.

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The kernel of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable and $\dim \ker L < \infty$, then the general solution is

$$\mathbf{x}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k,$$

where $\mathbf{x}_0$ is a particular solution, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for the kernel of L, and $t_1, \ldots, t_k$ are arbitrary scalars.

Example. $\begin{cases} x+y+z=4, \\ x+2y=3. \end{cases}$

$$L:\mathbb{R}^3 \to \mathbb{R}^2, \quad L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Linear equation: $L(\mathbf{x})=\mathbf{b}$, where $\mathbf{b}=\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} \color{red}{1} & 1 & 1 & 4 \\ 0 & \color{red}{1} & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} \color{red}{1} & 0 & 2 & 5 \\ 0 & \color{red}{1} & -1 & -1 \end{array}\right)$$

$$\begin{cases} x+2z=5 \\ y-z=-1 \end{cases} \iff \begin{cases} x=5-2z \\ y=-1+z \end{cases}$$

$$(x,y,z)=(5-2t,-1+t,t)=(5,-1,0)+t(-2,1,1).$$

Example. $u'''(x) - 2u''(x) + u'(x) = e^{2x}$.

Linear operator $L : C^3(\mathbb{R}) \to C(\mathbb{R})$,
$Lu = u''' - 2u'' + u'$.

Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

We already know that functions xe^x, e^x and 1 form a basis for the kernel of L. It remains to find a particular solution.

$L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x}$.
Since L is a linear operator, $L\left(\frac{1}{2}e^{2x}\right) = e^{2x}$.

Particular solution: $u_0(x) = \frac{1}{2}e^{2x}$.

Thus the general solution is

$$u(x) = \frac{1}{2}e^{2x} + t_1 xe^x + t_2 e^x + t_3.$$

Multiplication by a matrix as a linear map

Any $m \times n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $L(\mathbf{x}) \in \mathbb{R}^m$ are regarded as column vectors. This transformation is **linear**.

$$\textit{Example.} \quad L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ be the standard basis for $\mathbb{R}^3$. We have that $L(\mathbf{e}_1) = (1,3,0)$, $L(\mathbf{e}_2) = (0,4,5)$, $L(\mathbf{e}_3) = (2,7,8)$. Thus $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ are columns of the matrix.

Problem. Find a linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(\mathbf{e}_1) = (1,1)$, $L(\mathbf{e}_2) = (0,-2)$, $L(\mathbf{e}_3) = (3,0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbb{R}^3$.

$$L(x,y,z) = L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$$
$$= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3)$$
$$= x(1,1) + y(0,-2) + z(3,0) = (x+3z, x-2y)$$

$$L(x,y,z) = \begin{pmatrix} x+3z \\ x-2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{R}^n$.

$$\mathbf{y} = A\mathbf{x} \quad \Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Let V and W be vector spaces and S be a subset of V.

Theorem (i) If S spans V, then any linear transformation $L : V \to W$ is uniquely determined by its restriction to S.

(ii) If S is linearly independent then any function $L : S \to W$ can be extended to a linear transformation from V to W.

(iii) If S is a basis for V then any function $L : S \to W$ can be uniquely extended to a linear transformation from V to W.

Idea of the proof: If $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $\mathbf{v}_i \in S$, $r_i \in \mathbb{R}$, then $L(\mathbf{v}) = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2) + \cdots + r_nL(\mathbf{v}_n)$ for any linear map $L : V \to W$.