MATH 323

Linear Algebra

Lecture 24: Inner products. Orthogonality in inner product spaces.

Norm

The notion of *norm* generalizes the notion of length of a vector in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\alpha : V \to \mathbb{R}$, usually denoted $\alpha(\mathbf{x}) = \|\mathbf{x}\|$, is called a **norm** on V if it has the following properties:

(i) $\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\|r\mathbf{x}\| = |r| \, \|\mathbf{x}\|$ for all $r \in \mathbb{R}$ (homogeneity)
(iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

A **normed vector space** is a vector space endowed with a norm. The norm defines a distance function on the normed vector space: $\mathrm{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \dots, |x_n|)$.
- $\|\mathbf{x}\|_p = \big(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\big)^{1/p}$, $p \geq 1$.

Examples. $V = C[a,b]$, $f : [a,b] \to \mathbb{R}$.

- $\|f\|_\infty = \max\limits_{a \leq x \leq b} |f(x)|$.
- $\|f\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}$, $p \geq 1$.

Inner product

The notion of *inner product* generalizes the notion of dot product of vectors in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\beta : V \times V \to \mathbb{R}$, usually denoted $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$, is called an **inner product** on V if it is positive, symmetric, and bilinear. That is, if

(i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetry)
(iii) $\langle r\mathbf{x}, \mathbf{y} \rangle = r\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)
(iv) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (distributive law)

An **inner product space** is a vector space endowed with an inner product.

Examples. $V = \mathbb{R}^n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = d_1 x_1 y_1 + d_2 x_2 y_2 + \cdots + d_n x_n y_n$,

where $d_1, d_2, \dots, d_n > 0$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = (D\mathbf{x}) \cdot (D\mathbf{y})$,

where D is an invertible $n \times n$ matrix.

Remarks. (a) Invertibility of D is necessary to show that $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \implies \mathbf{x} = \mathbf{0}$.

(b) The second example is a particular case of the third one when $D = \mathrm{diag}(d_1^{1/2}, d_2^{1/2}, \dots, d_n^{1/2})$.

Problem. Find an inner product on $\mathbb{R}^2$ such that $\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 2$, $\langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 3$, and $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = -1$, where $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$.

Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$.
Then $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2$.
Using bilinearity, we obtain

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle x_1\mathbf{e}_1 + x_2\mathbf{e}_2, y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \rangle$$

$$= x_1\langle \mathbf{e}_1, y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \rangle + x_2\langle \mathbf{e}_2, y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \rangle$$

$$= x_1y_1\langle \mathbf{e}_1, \mathbf{e}_1 \rangle + x_1y_2\langle \mathbf{e}_1, \mathbf{e}_2 \rangle + x_2y_1\langle \mathbf{e}_2, \mathbf{e}_1 \rangle + x_2y_2\langle \mathbf{e}_2, \mathbf{e}_2 \rangle$$

$$= 2x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2.$$

It remains to check that $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ for $\mathbf{x} \neq \mathbf{0}$.
Indeed, $\langle \mathbf{x}, \mathbf{x} \rangle = 2x_1^2 - 2x_1x_2 + 3x_2^2 = (x_1 - x_2)^2 + x_1^2 + 2x_2^2$.

Example. $V = \mathcal{M}_{m,n}(\mathbb{R})$, space of $m \times n$ matrices.

- $\langle A, B \rangle = \mathrm{trace}\,(AB^T)$.

If $A = (a_{ij})$ and $B = (b_{ij})$, then $\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$.

Examples. $V = C[a, b]$.

- $\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$.

- $\langle f, g \rangle = \int_a^b f(x)g(x)w(x)\,dx$,

where w is bounded, piecewise continuous, and $w > 0$ everywhere on $[a, b]$.

w is called the **weight** function.

Theorem Suppose $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product on a vector space V. Then

$$\langle \mathbf{x}, \mathbf{y} \rangle^2 \le \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

Proof: For any $t \in \mathbb{R}$ let $\mathbf{v}_t = \mathbf{x} + t\mathbf{y}$. Then

$$\langle \mathbf{v}_t, \mathbf{v}_t \rangle = \langle \mathbf{x} + t\mathbf{y}, \mathbf{x} + t\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} + t\mathbf{y} \rangle + t\langle \mathbf{y}, \mathbf{x} + t\mathbf{y} \rangle$$
$$= \langle \mathbf{x}, \mathbf{x} \rangle + t\langle \mathbf{x}, \mathbf{y} \rangle + t\langle \mathbf{y}, \mathbf{x} \rangle + t^2\langle \mathbf{y}, \mathbf{y} \rangle.$$

Assume that $\mathbf{y} \neq \mathbf{0}$ and let $t = -\dfrac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$. Then

$$\langle \mathbf{v}_t, \mathbf{v}_t \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + t\langle \mathbf{y}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Since $\langle \mathbf{v}_t, \mathbf{v}_t \rangle \ge 0$, the desired inequality follows. In the case $\mathbf{y} = \mathbf{0}$, we have $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = 0$.

Cauchy-Schwarz Inequality:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}\,\sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Corollary 1 $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Equivalently, for all $x_i, y_i \in \mathbb{R}$,

$$(x_1 y_1 + \cdots + x_n y_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Corollary 2 For any $f, g \in C[a, b]$,

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \int_a^b |f(x)|^2\,dx \cdot \int_a^b |g(x)|^2\,dx.$$

Norms induced by inner products

Theorem Suppose $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product on a vector space V. Then $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is a norm.

Proof: Positivity is obvious. Homogeneity:

$$\|r\mathbf{x}\| = \sqrt{\langle r\mathbf{x}, r\mathbf{x} \rangle} = \sqrt{r^2 \langle \mathbf{x}, \mathbf{x} \rangle} = |r| \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Triangle inequality (follows from Cauchy-Schwarz's):

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &\leq \langle \mathbf{x}, \mathbf{x} \rangle + |\langle \mathbf{x}, \mathbf{y} \rangle| + |\langle \mathbf{y}, \mathbf{x} \rangle| + \langle \mathbf{y}, \mathbf{y} \rangle \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \, \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2. \end{aligned}$$

Examples. • The length of a vector in $\mathbb{R}^n$,

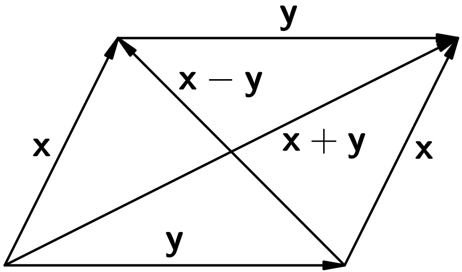
$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

is the norm induced by the dot product

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

• The norm $\|f\|_2 = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}$ on the vector space $C[a, b]$ is induced by the inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$



Parallelogram Identity:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

Proof: $\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle$.
Similarly, $\|\mathbf{x}-\mathbf{y}\|^2 = \langle \mathbf{x}, \mathbf{x}\rangle - \langle \mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle$.
Then $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{y}, \mathbf{y}\rangle = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$.

Remark. If a norm on a vector space fails to satisfy the Parallelogram Identity, it follows that the norm is not induced by any inner product.

Angle

Let V be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in V$ (the Cauchy-Schwarz inequality). Therefore we can define the **angle** between nonzero vectors in V by

$$\angle(\mathbf{x}, \mathbf{y}) = \arccos \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|}.$$

Then $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos \angle(\mathbf{x}, \mathbf{y})$.

In particular, vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Orthogonal sets

Let V be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. We say that vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Definition. A nonempty set $S \subset V$ of nonzero vectors is called an **orthogonal set** if all vectors in S are mutually orthogonal. That is, $\mathbf{0} \notin S$ and $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for any $\mathbf{x}, \mathbf{y} \in S$, $\mathbf{x} \neq \mathbf{y}$.
An orthogonal set $S \subset V$ is called **orthonormal** if $\|\mathbf{x}\| = 1$ for any $\mathbf{x} \in S$.

Theorem Any orthogonal set is linearly independent.

Example

- $V = C[-\pi, \pi]$, $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$.

$f_1(x) = \sin x$, $f_2(x) = \sin 2x$, $\dots$, $f_n(x) = \sin nx$, $\dots$

$$\langle f_m, f_n \rangle = \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } \ m = n, \\ 0 & \text{if } \ m \neq n. \end{cases}$$

Thus the set $\{f_1, f_2, f_3, \dots\}$ is orthogonal but not orthonormal.

It is orthonormal with respect to a scaled inner product

$$\langle\langle f, g \rangle\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

Orthonormal basis

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthonormal basis for an inner product space V.

Theorem 1 Let $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$ and $\mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n$, where $x_i, y_j \in \mathbb{R}$. Then

(i) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$,

(ii) $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

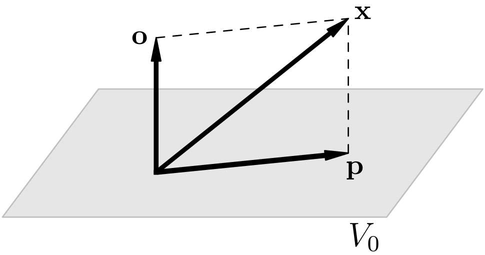
Theorem 2 For any vector $\mathbf{x} \in V$,

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{x}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Orthogonal projection

Theorem Let V be an inner product space and V_0 be a finite-dimensional subspace of V. Then any vector $\mathbf{x} \in V$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V_0$ and $\mathbf{o} \perp V_0$.

The component $\mathbf{p}$ is called the **orthogonal projection** of the vector $\mathbf{x}$ onto the subspace V_0.



The projection $\mathbf{p}$ is closer to $\mathbf{x}$ than any other vector in V_0. Hence the distance from $\mathbf{x}$ to V_0 is $\|\mathbf{x} - \mathbf{p}\| = \|\mathbf{o}\|$.

Theorem Let V be an inner product space and V_0 be a finite-dimensional subspace of V. Then any vector $\mathbf{x} \in V$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V_0$ and $\mathbf{o} \perp V_0$.

Theorem Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for the subspace V_0. Then for any vector $\mathbf{x} \in V$ the orthogonal projection $\mathbf{p}$ onto V_0 is given by

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n.$$

The Gram-Schmidt orthogonalization process

Let V be a vector space with an inner product. Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for V. Let

$$\mathbf{v}_1 = \mathbf{x}_1,$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \cdots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle}\mathbf{v}_{n-1}.$$

Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for V.

Normalization

Let V be a vector space with an inner product.
Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for V.
Let $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \dfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$,..., $\mathbf{w}_n = \dfrac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$.
Then $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ is an orthonormal basis for V.

Theorem Any finite-dimensional vector space with an inner product has an orthonormal basis.

Remark. An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.

Problem. Approximate the function $f(x) = e^x$ on the interval $[-1, 1]$ by a quadratic polynomial.

The best approximation would be a polynomial $p(x)$ that minimizes the distance relative to the uniform norm:

$$\|f - p\|_\infty = \max_{|x|\leq 1} |f(x) - p(x)|.$$

However there is no analytic way to find such a polynomial. Instead, one can find a **"least squares"** approximation that minimizes the integral norm

$$\|f - p\|_2 = \left(\int_{-1}^{1} |f(x) - p(x)|^2 \, dx \right)^{1/2}.$$

The norm $\|\cdot\|_2$ is induced by the inner product

$$\langle g, h\rangle = \int_{-1}^{1} g(x)h(x)\,dx.$$

Therefore $\|f - p\|_2$ is minimal if p is the orthogonal projection of the function f on the subspace $\mathcal{P}_3$ of quadratic polynomials.

We should apply the Gram-Schmidt process to the polynomials $1, x, x^2,$ which form a basis for $\mathcal{P}_3$. This would yield an orthogonal basis p_0, p_1, p_2. Then

$$p(x) = \frac{\langle f, p_0\rangle}{\langle p_0, p_0\rangle}p_0(x) + \frac{\langle f, p_1\rangle}{\langle p_1, p_1\rangle}p_1(x) + \frac{\langle f, p_2\rangle}{\langle p_2, p_2\rangle}p_2(x).$$