MATH 323 Fall 2023

Sample problems for Test 1

(to be worked out during the review)

Problem 1 Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Problem 2 Let A be a square matrix such that $A^3 = O$.

(i) Prove that the matrix A is not invertible.
(ii) Prove that the matrix $A + I$ is invertible.

Problem 3 Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Problem 4 Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

Problem 5 Determine which of the following subsets of $\mathbb{R}^\infty$ are subspaces. Briefly explain.

(i) The set S_1 of all arithmetic progressions.
(ii) The set S_2 of all geometric progressions.
(iii) The set S_3 of all square-summable sequences, i.e., sequences $(x_1, x_2, x_3, \dots)$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$.

Problem 6 Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Problem 7 Let V denote the solution set of a system

$$\begin{cases} x_2 + 2x_3 + 3x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0. \end{cases}$$

Find a basis for this subspace of $\mathbb{R}^4$, then extend it to a basis for $\mathbb{R}^4$.