MATH 409
Advanced Calculus I

Lecture 2: Properties of an ordered field. Absolute value. Supremum and infimum.

Real line

The real line is a mathematical object rich with structure. This includes:

- algebraic structure (4 arithmetic operations);
- ordering (for any three points, one is located between the other two);
- metric structure (we can measure distances between points);
- continuity (we can get from one point to another in a continuous way).

The algebraic structure is formalised by the notion of **field**. The ordering is formalised by the notion of **strict linear order**.

Field

A **field** is a set F equipped with two operations, **addition** $F \times F \ni (a, b) \mapsto a + b \in F$ and **multiplication** $F \times F \ni (a, b) \mapsto a \cdot b \in F$, such that:

F1. $a + b = b + a$ for all $a, b \in F$.
F2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in F$.
F3. There exists an element of F, denoted 0, such that $a + 0 = 0 + a = a$ for all $a \in F$.
F4. For any $a \in F$ there exists an element of F, denoted $-a$, such that $a + (-a) = (-a) + a = 0$.

F1′. $a \cdot b = b \cdot a$ for all $a, b \in F$.
F2′. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in F$.
F3′. There exists an element of F different from 0, denoted 1, such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in F$.
F4′. For any $a \in F$, $a \neq 0$ there exists an element of F, denoted a^{-1}, such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

F5. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in F$.

Alternative notation: $a \cdot b$ can be denoted ab (if it does not create confusion).

Auxiliary operations: **subtraction** $a - b = a + (-b)$ and **division** $a/b = a \cdot b^{-1}$.

Examples of fields:

- Real numbers $\mathbb{R}$.
- Complex numbers $\mathbb{C}$.
- Rational numbers $\mathbb{Q}$.
- $\mathbb{R}(x)$: rational functions $f(x)$ in variable x with real coefficients; $f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$, where $a_i, b_j \in \mathbb{R}$ and $b_m \neq 0$.
- $\mathbb{F}_2$: field of two elements.

Basic properties of fields

- The zero 0 and the unity 1 are unique.
- For any $a \in F$, the negative $-a$ is unique.
- For any $a \neq 0$, the inverse a^{-1} is unique.
- $-(-a) = a$ for all $a \in F$.
- $0 \cdot a = 0$ for any $a \in F$.
- $(-1) \cdot a = -a$ for any $a \in F$.
- $(-1) \cdot (-1) = 1$.
- $ab = 0$ implies that $a = 0$ or $b = 0$.
- $(a - b)c = ac - bc$ for all $a, b, c \in F$.

Strict linear order

Definition. A **strict order** on a set X is a relation on X, usually denoted $\prec$, that is antisymmetric and transitive, namely,

- $a \prec b \implies$ not $b \prec a$,
- $a \prec b$ and $b \prec c \implies a \prec c$.

The strict order $\prec$ is called **linear** (or **total**) if for any $a, b \in X$ we have either $a \prec b$ or $b \prec a$ or $a = b$.

Auxiliary notation. $a \succ b$ means that $b \prec a$.
By $a \preceq b$ we mean that $a \prec b$ or $a = b$.
By $a \prec b \prec c$ we mean that $a \prec b$ and $b \prec c$.

Ordered field

Definition. A field F with a strict linear order $\prec$ is called an **ordered field** if this order and arithmetic operations on F satisfy the following axioms:

OA. $a \prec b$ implies $a + c \prec b + c$,

OM1. $a \prec b$ and $c \succ 0$ imply $ac \prec bc$,

OM2. $a \prec b$ and $c \prec 0$ imply $ac \succ bc$.

Two axioms OM1 and OM2 can be replaced by one:

OM. $0 \prec a$ and $0 \prec b$ imply $0 \prec ab$.

Theorem Three axioms OA, OM1, and OM2 are equivalent to two axioms OA and OM.

OA. $a \prec b$ implies $a + c \prec b + c$ for all $a, b, c \in F$.
OM1. $a \prec b$ and $c \succ 0$ imply $ac \prec bc$ for all $a, b, c \in F$.
OM2. $a \prec b$ and $c \prec 0$ imply $ac \succ bc$ for all $a, b, c \in F$.
OM. $0 \prec a$ and $0 \prec b$ imply $0 \prec ab$ for all $a, b \in F$.

Theorem Three axioms OA, OM1, and OM2 are equivalent to two axioms OA and OM.

Proof: We have to prove that

$$\text{OA} \wedge \text{OM1} \wedge \text{OM2} \iff \text{OA} \wedge \text{OM},$$

where $\wedge$ denotes the logical operation "and". It is the same as to prove that $\text{OA} \wedge \text{OM1} \wedge \text{OM2} \implies \text{OA} \wedge \text{OM}$ and $\text{OA} \wedge \text{OM} \implies \text{OA} \wedge \text{OM1} \wedge \text{OM2}$.

[OA $\wedge$ OM1 $\wedge$ OM2 $\implies$ OM]
Assume that $0 \prec a$ and $0 \prec b$. Axiom OM1 implies that $0 \cdot b \prec ab$. We already know that $0 \cdot b = 0$. Thus $0 \prec ab$.

OA. $a \prec b$ implies $a + c \prec b + c$ for all $a, b, c \in F$.
OM1. $a \prec b$ and $c \succ 0$ imply $ac \prec bc$ for all $a, b, c \in F$.
OM2. $a \prec b$ and $c \prec 0$ imply $ac \succ bc$ for all $a, b, c \in F$.
OM. $0 \prec a$ and $0 \prec b$ imply $0 \prec ab$ for all $a, b \in F$.

Theorem Three axioms OA, OM1, and OM2 are equivalent to two axioms OA and OM.

Proof:

[OA $\wedge$ OM $\implies$ OM1] Assume that $a \prec b$ and $c \succ 0$. By Axiom OA, $a \prec b$ implies $a + (-a) \prec b + (-a)$, that is, $0 \prec b - a$. By Axiom OM, $0 \prec (b - a)c = bc - ac$. Adding ac to both sides of the latter relation, we get $ac \prec bc$.

[OA $\wedge$ OM $\implies$ OM2] Assume that $a \prec b$ and $c \prec 0$. By Axiom OA, $a \prec b$ implies $0 \prec b - a$ while $c \prec 0$ implies $0 \prec -c$. By Axiom OM, we get $0 \prec (b - a)(-c) = ac - bc$. Adding bc to both sides of the latter relation, we get $bc \prec ac$.

Properties of ordered fields

- $a \succ 0$ implies $-a \prec 0$.

Subtracting a from both sides of the relation $a \succ 0$, we get $0 \succ -a$.

- $a \prec b$ implies $a - b \prec 0$.

Subtracting b from both sides of $a \prec b$, we get $a - b \prec b - b = 0$.

- $a \prec b$ and $c \prec d$ imply $a + c \prec b + d$.

Adding c to both sides of $a \prec b$, we get $a + c \prec b + c$.
Adding b to both sides of $c \prec d$, we get $b + c \prec b + d$.
By transitivity of the order, $a + c \prec b + d$.

- $0 \prec a \prec b$ and $0 \prec c \prec d$ implies $ac \prec bd$.

Properties of ordered fields

- $a \succ 0$ and $b \prec 0$ imply $ab \prec 0$.

$b \prec 0$ implies $-b \succ 0$. Then $a(-b) \succ 0$. Note that $a(-b) = a(-1 \cdot b) = (-1)(ab) = -ab$. Hence $-ab \succ 0$ so that $ab \prec 0$.

- $a \prec 0$ and $b \prec 0$ imply $ab \succ 0$.

It follows that $-a \succ 0$ and $-b \succ 0$. Then $(-a)(-b) \succ 0$. But $(-a)(-b) = (-1 \cdot a)(-1 \cdot b) = (-1)(-1)ab = 1ab = ab$.

- $a \neq 0$ implies $a^2 \succ 0$ (where $a^2 = a \cdot a$).

Since $a \neq 0$, we have either $a \succ 0$ or $a \prec 0$. In the first case, $a^2 \succ 0$ due to Axiom OM. In the second case, $a^2 \succ 0$ by the previous property.

Properties of ordered fields

- $-1 \prec 0 \prec 1$.

Since $1 \neq 0$ and $a^2 \succ 0$ for any $a \neq 0$, we obtain $0 \prec 1^2 = 1$. Then $-1 \prec 0$.

- $0 \prec a$ implies $0 \prec a^{-1}$.

We know that either $0 \prec a^{-1}$ or $a^{-1} \prec 0$ or $a^{-1} = 0$. However $a^{-1} \prec 0$ would imply that $1 = aa^{-1} \prec 0$, a contradiction. Further, $a^{-1} = 0$ would imply that $1 = aa^{-1} = a \cdot 0 = 0$, another contradiction. Hence $0 \prec a^{-1}$.

- $0 \prec a \prec b$ implies $a^{-1} \succ b^{-1}$.

Since $0 \prec a$ and $0 \prec b$, it follows that $0 \prec a^{-1}$ and $0 \prec b^{-1}$. Multiplying both sides of $a \prec b$ by $a^{-1}b^{-1}$, we get $b^{-1} \prec a^{-1}$.

Which fields can be ordered?

- $\mathbb{R}$ is an ordered field with respect to the order $<$.
- $\mathbb{Q}$ is also an ordered field with respect to $<$.
- The field $\mathbb{F}_2$ of two elements cannot be ordered.

In any ordered field, $-1 \prec 0 \prec 1$, in particular, $-1 \prec 1$.
However in the field $\mathbb{F}_2$ we have $-1 = 1$.

- The field $\mathbb{C}$ cannot be ordered.

In any ordered field, $-1 \prec 0$ and $a^2 \succ 0$ for all $a \neq 0$.
However in the field $\mathbb{C}$ we have $i^2 = -1$, where
$i = \sqrt{-1} \neq 0$.

- The field $\mathbb{R}(x)$ of rational functions is an ordered field with respect to some strict linear order.

Absolute value

Definition. The **absolute value** (or **modulus**) of a real number a, denoted $|a|$, is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Properties of the absolute value:

- $|a| \geq 0$;
- $|a| = 0$ if and only if $a = 0$;
- $|-a| = |a|$;
- If $M > 0$, then $|a| < M \iff -M < a < M$;
- $|ab| = |a| \cdot |b|$;
- $|a + b| \leq |a| + |b|$.

Supremum and infimum

Definition. Let $E \subset \mathbb{R}$ be a nonempty set and M be a real number. We say that M is an **upper bound** of the set E if $a \leq M$ for all $a \in E$. Similarly, M is a **lower bound** of the set E if $a \geq M$ for all $a \in E$.

We say that the set E is **bounded above** if it admits an upper bound and **bounded below** if it admits a lower bound. The set E is called **bounded** if it is bounded above and below.

A real number M is called the **supremum** (or the **least upper bound**) of the set E and denoted $\sup E$ if (i) M is an upper bound of E and (ii) $M \leq M_+$ for any upper bound M_+ of E. Similarly, M is called the **infimum** (or the **greatest lower bound**) of the set E and denoted $\inf E$ if (i) M is a lower bound of E and (ii) $M \geq M_-$ for any lower bound M_- of E.

Completeness Axiom. If a nonempty subset $E \subset \mathbb{R}$ is bounded above, then E has a supremum.