MATH 409

Advanced Calculus I

Lecture 7: Monotone sequences. The Bolzano-Weierstrass theorem.

Limit of a sequence

Definition. Sequence $\{x_n\}$ of real numbers is said to **converge** to a real number a if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N$. The number a is called the **limit** of $\{x_n\}$.

A sequence is called **convergent** if it has a limit and **divergent** otherwise.

Properties of convergent sequences:

- the limit is unique;
- any convergent sequence is bounded;
- any subsequence of a convergent sequence converges to the same limit;
- modifying a finite number of elements cannot affect convergence of a sequence or change its limit;
- rearranging elements of a sequence cannot affect its convergence or change its limit.

Limit theorems

Theorem 1 If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$ and $x_n \leq w_n \leq y_n$ for all sufficiently large n, then $\lim_{n\to\infty} w_n = a$.

Theorem 2 If $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$, and $x_n \leq y_n$ for all sufficiently large n, then $a \leq b$.

Theorem 3 If $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$, then $\lim_{n\to\infty}(x_n + y_n) = a + b$, $\lim_{n\to\infty}(x_n - y_n) = a - b$, and $\lim_{n\to\infty} x_n y_n = ab$. If, additionally, $b \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n / y_n = a/b$.

Examples

- $$\lim_{n\to\infty} \frac{\sin(e^n)}{n} = 0.$$

$-1/n \leq \sin(e^n)/n \leq 1/n$ for all $n \in \mathbb{N}$ since $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$. As shown in the previous lecture, $1/n \to 0$ as $n \to \infty$. Then $-1/n \to -1 \cdot 0 = 0$ as $n \to \infty$. By the Squeeze Theorem, $\sin(e^n)/n \to 0$ as $n \to \infty$.

- $$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

The sequence $\{1/2^n\}$ is a subsequence of $\{1/n\}$. Hence it is converging to the same limit.

Examples

- $\displaystyle \lim_{n\to\infty} \frac{(1+2n)^2}{1+2n^2} = 2.$

$$\frac{(1+2n)^2}{1+2n^2} = \frac{(1+2n)^2/n^2}{(1+2n^2)/n^2} = \frac{(1/n+2)^2}{(1/n)^2+2} \quad \text{for all } n \in \mathbb{N}.$$

Since $1/n \to 0$ as $n \to \infty$, it follows that

$$1/n + 2 \to 0 + 2 = 2 \quad \text{as} \quad n \to \infty,$$

$$(1/n+2)^2 \to 2^2 = 4 \quad \text{as} \quad n \to \infty,$$

$$(1/n)^2 \to 0^2 = 0 \quad \text{as} \quad n \to \infty,$$

$$(1/n)^2 + 2 \to 0 + 2 = 2 \quad \text{as} \quad n \to \infty,$$

and, finally, $\displaystyle \frac{(1/n+2)^2}{(1/n)^2+2} \to \frac{4}{2} = 2$ as $n \to \infty$.

Monotone sequences

Definition. A sequence $\{x_n\}$ is called **increasing** (or **nondecreasing**) if $x_1 \leq x_2 \leq x_3 \leq \dots$ or, to be precise, $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. It is called **strictly increasing** if $x_1 < x_2 < x_3 < \dots$, that is, $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.

Likewise, the sequence $\{x_n\}$ is called **decreasing** (or **nonincreasing**) if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. It is called **strictly decreasing** if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.

Increasing and decreasing sequences are called **monotone**.

Examples:

- the sequence $\{1/n\}$ is strictly decreasing;
- the sequence $1, 1, 2, 2, 3, 3, \dots$ is increasing, but not strictly increasing;
- the sequence $-1, 1, -1, 1, -1, 1, \dots$ is neither increasing nor decreasing;
- a constant sequence is both increasing and decreasing.

Theorem Any increasing sequence converges to a limit if it is bounded, and diverges to $+\infty$ otherwise.

Proof: Let $\{x_n\}$ be an increasing sequence. First consider the case when $\{x_n\}$ is bounded. In this case, the set E of all elements occurring in the sequence is bounded. Then $\sup E$ exists. We claim that $x_n \to \sup E$ as $n \to \infty$. Take any $\varepsilon > 0$. Then $\sup E - \varepsilon$ is not an upper bound of E. Hence there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} > \sup E - \varepsilon$. Since the sequence is increasing, it follows that $x_n \geq x_{n_0} > \sup E - \varepsilon$ for all $n \geq n_0$. At the same time, $x_n \leq \sup E$ for all $n \in \mathbb{N}$. Therefore $|x_n - \sup E| < \varepsilon$ for all $n \geq n_0$, which proves the claim.

Now consider the case when the sequence $\{x_n\}$ is not bounded. Note that the set E is bounded below (as x_1 is a lower bound). Hence E is not bounded above. Then for any $C \in \mathbb{R}$ there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} > C$. It follows that $x_n \geq x_{n_0} > C$ for all $n \geq n_0$. Thus $\{x_n\}$ diverges to $+\infty$.

Theorem Any decreasing sequence converges to a limit if it is bounded, and diverges to $-\infty$ otherwise.

Proof: Let $\{x_n\}$ be a decreasing sequence. Then the sequence $\{-x_n\}$ is increasing since the inequality $a \geq b$ is equivalent to $-a \leq -b$ for all $a, b \in \mathbb{R}$. By the previous theorem, either $-x_n \to c$ for some $c \in \mathbb{R}$ as $n \to \infty$, or else $-x_n$ diverges to $+\infty$. In the former case, $x_n \to -c$ as $n \to \infty$ (in particular, it is bounded). In the latter case, x_n diverges to $-\infty$ (in particular, it is unbounded).

Corollary Any monotone sequence converges to a limit if it is bounded, and diverges to infinity otherwise.

Nested intervals property

Definition. A sequence of sets $I_1, I_2, \dots$ is called **nested** if $I_1 \supset I_2 \supset \dots$, that is, $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$.

Theorem If $\{I_n\}$ is a nested sequence of nonempty closed bounded intervals, then the intersection $\bigcap_{n \in \mathbb{N}} I_n$ is nonempty. Moreover, if lengths $|I_n|$ of the intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then the intersection consists of a single point.

Remark 1. The theorem may not hold if the intervals $I_1, I_2, \dots$ are open. Counterexample: $I_n = (0, 1/n)$, $n \in \mathbb{N}$. The intervals are nested and bounded, but their intersection is empty since $1/n \to 0$ as $n \to \infty$.

Remark 2. The theorem may not hold if the intervals $I_1, I_2, \dots$ are not bounded. Counterexample: $I_n = [n, \infty)$, $n \in \mathbb{N}$. The intervals are nested and closed, but their intersection is empty since the sequence $\{n\}$ diverges to $+\infty$.

Proof of the theorem

Let $I_n = [a_n, b_n]$, $n = 1, 2, \dots$ Since the sequence $\{I_n\}$ is nested, it follows that the sequence $\{a_n\}$ is increasing while the sequence $\{b_n\}$ is decreasing. Besides, both sequences are bounded (since both are contained in the interval I_1). Hence both are convergent: $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Since $a_n \leq b_n$ for all $n \in \mathbb{N}$, the Comparison Theorem implies that $a \leq b$. We claim that $\bigcap_{n\in\mathbb{N}} I_n = [a, b]$. Indeed, we have $a_n \leq a$ for all $n \in \mathbb{N}$ (by the Comparison Theorem applied to $a_1, a_2, \dots$ and the constant sequence $a_n, a_n, a_n \dots$). Similarly, $b \leq b_n$ for all $n \in \mathbb{N}$. Therefore $[a, b]$ is contained in the intersection. On the other hand, if $x < a$ then $x < a_n$ for some n so that $x \notin I_n$. Similarly, if $x > b$ then $x > b_m$ for some m so that $x \notin I_m$. This proves the claim.

Clearly, the length of $[a, b]$ cannot exceed $|I_n|$ for any $n \in \mathbb{N}$. Therefore $|I_n| \to 0$ as $n \to \infty$ implies that $[a, b]$ is a degenerate interval: $a = b$.

Bolzano-Weierstrass Theorem

Theorem Every bounded sequence of real numbers has a convergent subsequence.

Proof: Let $\{x_n\}$ be a bounded sequence of real numbers. We are going to build a nested sequence of intervals $I_n = [a_n, b_n]$, $n = 1, 2, \dots$, such that each I_n contains infinitely many elements of $\{x_n\}$ and $|I_{n+1}| = |I_n|/2$ for all $n \in \mathbb{N}$. The sequence is built inductively. First we set I_1 to be any closed bounded interval that contains all elements of $\{x_n\}$ (such an interval exists since the sequence $\{x_n\}$ is bounded). Now assume that for some $n \in \mathbb{N}$ the interval I_n is already chosen and it contains infinitely many elements of the sequence $\{x_n\}$. Then at least one of the subintervals $I' = [a_n, (a_n + b_n)/2)]$ and $I'' = [(a_n + b_n)/2, b_n]$ also contains infinitely many elements of $\{x_n\}$. We set I_{n+1} to be such a subinterval. By construction, $I_{n+1} \subset I_n$ and $|I_{n+1}| = |I_n|/2$.

Proof (continued): Since $|I_{n+1}| = |I_n|/2$ for all $n \in \mathbb{N}$, it follows by induction that $|I_n| = |I_1|/2^{n-1}$ for all $n \in \mathbb{N}$. As a consequence, $|I_n| \to 0$ as $n \to \infty$. By the nested intervals property, the intersection of the intervals $I_1, I_2, I_3, \dots$ consists of a single number a.

Next we are going to build a strictly increasing sequence of natural numbers $n_1, n_2, \dots$ such that $x_{n_k} \in I_k$ for all $k \in \mathbb{N}$. The sequence is built inductively. First we choose n_1 so that $x_{n_1} \in I_1$. Now assume that for some $k \in \mathbb{N}$ the number n_k is already chosen. Since the interval I_{k+1} contains infinitely many elements of the sequence $\{x_n\}$, there exists $m > n_k$ such that $x_m \in I_{k+1}$. We set $n_{k+1} = m$.

Now we claim that the subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of the sequence $\{x_n\}$ converges to a. Indeed, for any $k \in \mathbb{N}$ the points x_{n_k} and a both belong to the interval I_k. Hence $|x_{n_k} - a| \le |I_k|$. Since $|I_k| \to 0$ as $k \to \infty$, it follows that $x_{n_k} \to a$ as $k \to \infty$.