MATH 409

Advanced Calculus I

Lecture 12: Uniform continuity. Exponential functions.

Uniform continuity

Definition. A function $f : E \to \mathbb{R}$ defined on a set $E \subset \mathbb{R}$ is called **uniformly continuous** on E if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - y| < \delta$ and $x, y \in E$ imply $|f(x) - f(y)| < \varepsilon$.

Recall that the function f is continuous at a point $y \in E$ if for every $\varepsilon > 0$ there exists $\delta = \delta(y, \varepsilon) > 0$ such that $|x - y| < \delta$ and $x \in E$ imply $|f(x) - f(y)| < \varepsilon$.

Therefore the uniform continuity of f is a stronger property than the continuity of f on E.

Examples

- Constant function $f(x) = a$ is uniformly continuous on $\mathbb{R}$.

Indeed, $|f(x) - f(y)| = 0 < \varepsilon$ for any $\varepsilon > 0$ and $x, y \in \mathbb{R}$.

- Identity function $f(x) = x$ is uniformly continuous on $\mathbb{R}$.

Since $f(x) - f(y) = x - y$, we have $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \varepsilon$.

- The sine function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

It was shown in the previous lecture that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Therefore $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \varepsilon$.

Lipschitz functions

Definition. A function $f : E \to \mathbb{R}$ is called a **Lipschitz function** if there exists a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in E$.

- Any Lipschitz function is uniformly continuous.

Using notation of the definition, let $\delta(\varepsilon) = \varepsilon/L$, $\varepsilon > 0$.
Then $|x - y| < \delta(\varepsilon)$ implies

$$|f(x) - f(y)| \leq L|x - y| < L\delta(\varepsilon) = \varepsilon$$

for all $x, y \in E$.

- The function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ but not Lipschitz.

For any $n \in \mathbb{N}$, $|f(1/n) - f(0)| = \sqrt{1/n} = \sqrt{n}\,|1/n - 0|$.
It follows that f is not Lipschitz.

Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. Suppose $|x - y| < \delta$, where $x, y \geq 0$. To estimate $|f(x) - f(y)|$, we consider two cases.

In the case $x, y \in [0, \delta)$, we use the fact that f is strictly increasing. Then $|f(x) - f(y)| < f(\delta) - f(0) = \sqrt{\delta} = \varepsilon$.

Otherwise, when $x \notin [0, \delta)$ or $y \notin [0, \delta)$, we have $\max(x, y) \geq \delta$. Then

$$|\sqrt{x} - \sqrt{y}| = \left| \frac{x - y}{\sqrt{x} + \sqrt{y}} \right| \leq \frac{|x - y|}{\sqrt{\max(x, y)}} < \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta} = \varepsilon.$$

Thus f is uniformly continuous.

- The function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Let $\varepsilon = 2$ and choose an arbitrary $\delta > 0$. Let n_δ be a natural number such that $1/n_\delta < \delta$. Further, let $x_\delta = n_\delta + 1/n_\delta$ and $y_\delta = n_\delta$. Then $|x_\delta - y_\delta| = 1/n_\delta < \delta$ while

$$f(x_\delta) - f(y_\delta) = (n_\delta + 1/n_\delta)^2 - n_\delta^2 = 2 + 1/n_\delta^2 > \varepsilon.$$

We conclude that f is not uniformly continuous.

- The function $f(x) = x^2$ is Lipschitz (and hence uniformly continuous) on any bounded interval $[a, b]$.

For any $x, y \in [a, b]$ we obtain

$$\begin{aligned} |x^2 - y^2| &= |(x+y)(x-y)| = |x+y|\,|x-y| \\ &\le (|x| + |y|)\,|x-y| \le 2\max(|a|, |b|)\,|x-y|. \end{aligned}$$

Theorem Any function continuous on a closed bounded interval $[a, b]$ is also uniformly continuous on $[a, b]$.

Proof: Assume that a function $f : [a, b] \to \mathbb{R}$ is not uniformly continuous on $[a, b]$. We have to show that f is not continuous on $[a, b]$. By assumption, there exists $\varepsilon > 0$ such that for any $\delta > 0$ we can find two points $x, y \in [a, b]$ satisfying $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$. In particular, for any $n \in \mathbb{N}$ there exist points $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < 1/n$ while $|f(x_n) - f(y_n)| \geq \varepsilon$.

By construction, $\{x_n\}$ is a bounded sequence. According to the Bolzano-Weierstrass theorem, there is a subsequence $\{x_{n_k}\}$ converging to a limit c. Moreover, c belongs to $[a, b]$ as $\{x_n\} \subset [a, b]$. Since $x_n - 1/n < y_n < x_n + 1/n$ for all $n \in \mathbb{N}$, the subsequence $\{y_{n_k}\}$ also converges to c. However the inequalities $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$ imply that at least one of the sequences $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ is not converging to $f(c)$. It follows that the function f is not continuous at c.

Theorem Suppose that a function $f : E \to \mathbb{R}$ is uniformly continuous on E. Then it maps Cauchy sequences to Cauchy sequences, that is, for any Cauchy sequence $\{x_n\} \subset E$ the sequence $\{f(x_n)\}$ is also Cauchy.

Proof: Let $\{x_n\} \subset E$ be a Cauchy sequence. Since the function f is uniformly continuous on E, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $|x - y| < \delta$ and $x, y \in E$ imply $|f(x) - f(y)| < \varepsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists $N = N(\delta) \in \mathbb{N}$ such that $|x_n - x_m| < \delta$ for all $n, m \geq N$. Then $|f(x_n) - f(x_m)| < \varepsilon$ for all $n, m \geq N$. We conclude that $\{f(x_n)\}$ is a Cauchy sequence.

Dense subsets

Definition. Given a set $E \subset \mathbb{R}$ and its subset $E_0 \subset E$, we say that E_0 is **dense in** E if for any point $x \in E$ and any $\varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon)$ contains an element of E_0.

Examples. • An open bounded interval (a, b) is dense in the closed interval $[a, b]$.
• The set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$.

Theorem A subset E_0 of a set $E \subset \mathbb{R}$ is dense in E if and only if for any $c \in E$ there exists a sequence $\{x_n\} \subset E_0$ converging to c.

Proof: Suppose that for any point $c \in E$ there is a sequence $\{x_n\} \subset E_0$ converging to c. Then any ε-neighborhood $(c - \varepsilon, c + \varepsilon)$ of c contains an element of that sequence. Conversely, suppose that E_0 is dense in E. Then, given $c \in E$, for any $n \in \mathbb{N}$ there is a point $x_n \in (c - \frac{1}{n}, c + \frac{1}{n}) \cap E_0$. Clearly, $x_n \to c$ as $n \to \infty$.

Continuous extension

Theorem Suppose that a subset E_0 of a set $E \subset \mathbb{R}$ is dense in E. Then any uniformly continuous function $f : E_0 \to \mathbb{R}$ can be extended to a continuous function on E. Moreover, the extension is unique and uniformly continuous.

Proof: First let us show that a continuous extension of the function f to the set E is unique (assuming it exists).

Suppose $g, h : E \to \mathbb{R}$ are two continuous extensions of f. Since the set E_0 is dense in E, for any $c \in E$ there is a sequence $\{x_n\} \subset E_0$ converging to c. Since g and h are continuous at c, we get $g(x_n) \to g(c)$ and $h(x_n) \to h(c)$ as $n \to \infty$. However $g(x_n) = h(x_n) = f(x_n)$ for all $n \in \mathbb{N}$. Hence $g(c) = h(c)$.

Proof (continued): Given $c \in E$, let $\{x_n\}$ be a sequence of elements of E_0 converging to c. The sequence $\{x_n\}$ is Cauchy. Since f is uniformly continuous, it follows that the sequence $\{f(x_n)\}$ is also Cauchy. Hence it converges to a limit L. We claim that the limit L depends only on c and does not depend on the choice of the sequence $\{x_n\}$. Indeed, let $\{\tilde{x}_n\} \subset E_0$ be another sequence converging to c. Then a sequence $x_1, \tilde{x}_1, x_2, \tilde{x}_2, \dots$ also converges to c. Consequently, the sequence $f(x_1), f(\tilde{x}_1), f(x_2), f(\tilde{x}_2), \dots$ is convergent. The limit is L since the subsequence $\{f(x_n)\}$ converges to L. Another subsequence is $\{f(\tilde{x}_n)\}$, hence it converges to L as well. Now we set $F(c) = L$, which defines a function $F : E \to \mathbb{R}$.

The continuity of the function f implies that $F(c) = f(c)$ for $c \in E_0$, i.e., F is an extension of f.

Proof (continued): It remains to show that the extension F is uniformly continuous.

Given $\varepsilon > 0$, let $\varepsilon_0 = \varepsilon/2$. Since f is uniformly continuous, there is $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon_0$ for all $x, y \in E_0$. For any $c, d \in E$ we can find sequences $\{x_n\}$ and $\{y_n\}$ of elements of E_0 such that $x_n \to c$ and $y_n \to d$ as $n \to \infty$. By construction of F, we have $f(x_n) \to F(c)$ and $f(y_n) \to F(d)$ as $n \to \infty$.

If $|c - d| < \delta$, then $|x_n - y_n| < \delta$ for all sufficiently large n. Consequently, $|f(x_n) - f(y_n)| < \varepsilon_0$ for all sufficiently large n, which implies $|F(c) - F(d)| \leq \varepsilon_0 < \varepsilon$.

Thus F is uniformly continuous.

Exponential functions

Theorem For any $a > 0$ there exists a unique function $F_a : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(i) $F_a(1) = a$,
(ii) $F_a(x + y) = F_a(x)F_a(y)$ for all $x, y \in \mathbb{R}$,
(iii) F_a is continuous at 0.

Remark. The function is denoted $F_a(x) = a^x$ and called the **exponential function with base** a.

Sketch of the proof (existence)

Let $a^0 = 1$, $a^1 = a$, and $a^{n+1} = a^n a$ for all $n \in \mathbb{N}$.
Further, let $a^{-n} = 1/a^n$ for all $n \in \mathbb{N}$.

Lemma 1 $a^{m+n} = a^m a^n$ and $a^{mn} = (a^m)^n$ for all $m, n \in \mathbb{Z}$.

Lemma 2 If $m_1, m_2 \in \mathbb{Z}$ and $n_1, n_2 \in \mathbb{N}$ satisfy $m_1/n_1 = m_2/n_2$, then $\sqrt[n_1]{a^{m_1}} = \sqrt[n_2]{a^{m_2}}$.

For any $r \in \mathbb{Q}$ let $a^r = \sqrt[n]{a^m}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ are chosen so that $r = m/n$.

Lemma 3 $a^{r+s} = a^r a^s$ and $a^{rs} = (a^r)^s$ for all $r, s \in \mathbb{Q}$.

Lemma 4 The function $f(r) = a^r$, $r \in \mathbb{Q}$, is monotone.

Lemma 5 $a^{1/n} \to 1$ as $n \to \infty$.

Lemma 6 The function $f(r) = a^r$, $r \in \mathbb{Q}$, is uniformly continuous on $[b_1, b_2] \cap \mathbb{Q}$ for any bounded interval $[b_1, b_2]$.