MATH 409

Advanced Calculus I

# Lecture 13:
# Review for Test 1.

# Topics for Test 1

*Part I: Axiomatic model of the real numbers*

- Axioms of an ordered field
- Completeness axiom
- Archimedean principle
- Principle of mathematical induction
- Binomial formula
- Countable and uncountable sets

*Wade's book: 1.1–1.6, Appendix A*

## Topics for Test 1

*Part II: Limits and continuity*

- Limits of sequences
- Limit theorems for sequences
- Monotone sequences
- Bolzano-Weierstrass theorem
- Cauchy sequences
- Limits of functions
- Limit theorems for functions
- Continuity of functions
- Extreme value and intermediate value theorems
- Uniform continuity

*Wade's book: 2.1–2.5, 3.1–3.4*

# Axioms of real numbers

*Definition.* The set $\mathbb{R}$ of real numbers is a set satisfying the following postulates:

**Postulate 1.** $\mathbb{R}$ is a field.

**Postulate 2.** There is a strict linear order $<$ on $\mathbb{R}$ that makes it into an ordered field.

**Postulate 3 (Completeness Axiom).**
If a nonempty subset $E \subset \mathbb{R}$ is bounded above, then $E$ has a supremum.

# Theorems to know

**Theorem (Archimedean Principle)** For any real number $\varepsilon > 0$ there exists a natural number $n$ such that $n\varepsilon > 1$.

**Theorem (Principle of mathematical induction)** Let $P(n)$ be an assertion depending on a natural variable $n$. Suppose that

- $P(1)$ holds,
- whenever $P(k)$ holds, so does $P(k+1)$.

Then $P(n)$ holds for all $n \in \mathbb{N}$.

**Theorem** If $A_1, A_2, \ldots$ are finite or countable sets, then the union $A_1 \cup A_2 \cup \ldots$ is also finite or countable. As a consequence, the sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{N} \times \mathbb{N}$ are countable.

**Theorem** The set $\mathbb{R}$ is uncountable.

## Limit theorems for sequences

**Theorem** If $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = a$ and $x_n \le w_n \le y_n$ for all sufficiently large $n$, then $\lim\limits_{n\to\infty} w_n = a$.

**Theorem** If $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$, and $x_n \le y_n$ for all sufficiently large $n$, then $a \le b$.

**Theorem** If $\lim\limits_{n\to\infty} x_n = a$ and $\lim\limits_{n\to\infty} y_n = b$, then $\lim\limits_{n\to\infty}(x_n + y_n) = a + b$, $\lim\limits_{n\to\infty}(x_n - y_n) = a - b$, and $\lim\limits_{n\to\infty} x_n y_n = ab$. If, additionally, $b \ne 0$ and $y_n \ne 0$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} x_n/y_n = a/b$.

**Theorem** Any monotone sequence converges to a limit if bounded, and diverges to infinity otherwise.

**Theorem (Bolzano-Weierstrass)** Every bounded sequence of real numbers has a convergent subsequence.

**Theorem** Any Cauchy sequence is convergent.

**Theorem** A function $f : E \to \mathbb{R}$ is continuous at a point $c \in E$ if and only if for any sequence $\{x_n\}$ of elements of $E$, $x_n \to c$ as $n \to \infty$ implies $f(x_n) \to f(c)$ as $n \to \infty$.

**Theorem** Suppose that functions $f, g : E \to \mathbb{R}$ are both continuous at a point $c \in E$. Then the functions $f + g$, $f - g$, and $fg$ are also continuous at $c$. If, additionally, $g(c) \neq 0$, then the function $f/g$ is continuous at $c$ as well.

**Extreme Value Theorem** If $I = [a, b]$ is a closed, bounded interval of the real line, then any continuous function $f : I \to \mathbb{R}$ is bounded and attains its extreme values (maximum and minimum) on $I$.

**Intermediate Value Theorem** If a function $f : [a, b] \to \mathbb{R}$ is continuous then any number $y_0$ that lies between $f(a)$ and $f(b)$ is a value of $f$, i.e., $y_0 = f(x_0)$ for some $x_0 \in [a, b]$.

**Theorem** Any function continuous on a closed bounded interval $[a, b]$ is also uniformly continuous on $[a, b]$.

# Sample problems for Test 1

**Problem 1 (15 pts.)** Prove that for any $n \in \mathbb{N}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

**Problem 2 (30 pts.)** Let $\{F_n\}$ be the sequence of Fibonacci numbers: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

**(i)** Show that the sequence $\{F_{2k}/F_{2k-1}\}_{k \in \mathbb{N}}$ is increasing while the sequence $\{F_{2k+1}/F_{2k}\}_{k \in \mathbb{N}}$ is decreasing.

**(ii)** Prove that $\displaystyle\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2}.$

# Sample problems for Test 1

**Problem 3 (25 pts.)** Prove the Extreme Value Theorem: if $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed bounded interval $[a, b]$, then $f$ is bounded and attains its extreme values (maximum and minimum) on $[a, b]$.

# Sample problems for Test 1

**Problem 4 (20 pts.)** Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(-1) = f(0) = f(1) = 0$ and $f(x) = \dfrac{x-1}{x^2-1} \sin \dfrac{1}{x}$ for $x \in \mathbb{R} \setminus \{-1, 0, 1\}$.

**(i)** Determine all points at which the function $f$ is continuous.

**(ii)** Is the function $f$ uniformly continuous on the interval $(0, 1)$? Is it uniformly continuous on the interval $(1, 2)$? Explain.

# Sample problems for Test 1

**Bonus Problem 5 (15 pts.)** Given a set $X$, let $\mathcal{P}(X)$ denote the set of all subsets of $X$. Prove that $\mathcal{P}(X)$ is not of the same cardinality as $X$.

**Problem 1.** Prove that for any $n \in \mathbb{N}$,
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

*Proof:* The proof is by induction on $n$. First we consider the case $n = 1$. In this case the formula reduces to $1^3 = \frac{1^2 \cdot 2^2}{4}$, which is a true equality. Now assume that the formula holds for $n = k$, that is,
$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$
Adding $(k+1)^3$ to both sides of this equality, we get
$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$
$$= (k+1)^2 \left( \tfrac{k^2}{4} + (k+1) \right) = (k+1)^2 \, \tfrac{k^2+4k+4}{4} = \tfrac{(k+1)^2(k+2)^2}{4},$$
which means that the formula holds for $n = k+1$ as well. By induction, the formula holds for any natural number $n$.

*Remark.* We have proved that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Also, it is known that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

It follows that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

for all $n \in \mathbb{N}$.

**Problem 2.** Let $\{F_n\}$ be the sequence of Fibonacci numbers: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
**(i)** Show that the sequence $\{F_{2k}/F_{2k-1}\}_{k\in\mathbb{N}}$ is increasing while the sequence $\{F_{2k+1}/F_{2k}\}_{k\in\mathbb{N}}$ is decreasing.

Let $x_n = F_{n+1}/F_n$, $n \in \mathbb{N}$. Then

$$x_{n+1} = \frac{F_{n+2}}{F_{n+1}} = \frac{F_n + F_{n+1}}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}} = 1 + \frac{1}{x_n}$$

for all $n \in \mathbb{N}$. In particular, $x_1 = 1$, $x_2 = 1 + 1/x_1 = 2$, $x_3 = 1 + 1/x_2 = 3/2$, $x_4 = 1 + 1/x_3 = 5/3$. Notice that

$$x_1 < x_3 < x_4 < x_2.$$

The function $f(x) = 1 + 1/x$ is strictly decreasing on the interval $I = (0, \infty)$ and maps it to itself. Therefore its second iteration $g = f \circ f$ is strictly increasing on $I$ and $g(I) \subset I$. We have $x_{n+2} = f(x_{n+1}) = f(f(x_n)) = g(x_n)$ for all $n \in \mathbb{N}$. Now it follows by induction on $k$ that

$$x_{2k-1} < x_{2k+1} < x_{2k+2} < x_{2k} \quad \text{for all } k \in \mathbb{N}.$$

**Problem 2.** Let $\{F_n\}$ be the sequence of Fibonacci numbers: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

**(ii)** Prove that $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n} = \dfrac{\sqrt{5}+1}{2}$.

We already know that the numbers $x_n = F_{n+1}/F_n$ satisfy inequalities

$$x_{2k-1} < x_{2k+1} < x_{2k+2} < x_{2k}$$

for all $k \in \mathbb{N}$. It follows that the sequence $\{x_{2k-1}\}$ is strictly increasing, the sequence $\{x_{2k}\}$ is strictly decreasing, and both sequences are bounded. Therefore these sequences are converging to some positive limits: $x_{2k-1} \to c_1$ and $x_{2k} \to c_2$ as $k \to \infty$. To prove that $\lim\limits_{n\to\infty} F_{n+1}/F_n = (\sqrt{5}+1)/2$, it is enough to show that $c_1 = c_2 = (\sqrt{5}+1)/2$.

For any $x > 0$ we obtain

$$g(x) = f(f(x)) = f\left(1 + \frac{1}{x}\right) = 1 + \frac{1}{1 + \frac{1}{x}}$$

$$= 1 + \frac{1}{\frac{x+1}{x}} = 1 + \frac{x}{x+1} = \frac{2x+1}{x+1}.$$

It follows that $g(x_{2k-1}) \to g(c_1)$ and $g(x_{2k}) \to g(c_2)$ as $k \to \infty$. However $g(x_{2k-1}) = x_{2k+1}$ and $g(x_{2k}) = x_{2k+2}$, which implies that $g(c_1) = c_1$ and $g(c_2) = c_2$. Since

$$x - g(x) = \frac{x(x+1)}{x+1} - \frac{2x+1}{x+1} = \frac{x^2 - x - 1}{x+1},$$

$c_1$ and $c_2$ are roots of the equation $x^2 - x - 1 = 0$. This equation has two roots, $(1 - \sqrt{5})/2$ and $(\sqrt{5} + 1)/2$. One of the roots is negative. Thus both $c_1$ and $c_2$ are equal to the other root, $(\sqrt{5} + 1)/2$.

**Problem 3.** Prove the Extreme Value Theorem: if $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed bounded interval $[a, b]$, then $f$ is bounded and attains its extreme values (maximum and minimum) on $[a, b]$.

*Proof:* First let us prove that the function $f$ is bounded. Assume the contrary. Then for every $n \in \mathbb{N}$ there exists a point $x_n \in [a, b]$ such that $|f(x_n)| > n$. We obtain a sequence $\{x_n\}$ of elements of $[a, b]$ such that the sequence $\{f(x_n)\}$ diverges to infinity. Since the sequence $\{x_n\}$ is bounded, it has a convergent subsequence $\{x_{n_k}\}$ due to the Bolzano-Weierstrass Theorem. Let $c$ be the limit of $x_{n_k}$ as $k \to \infty$. Since $a \leq x_{n_k} \leq b$ for all $k$, the Comparison Theorem implies that $a \leq c \leq b$, i.e., $c \in [a, b]$. Then the function $f$ is continuous at $c$. As a consequence, $f(x_{n_k}) \to f(c)$ as $k \to \infty$. However the sequence $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$ and hence diverges to infinity. This contradiction shows that the assumption was wrong: the function $f$ is bounded.

Since the function $f$ is bounded, the image $f([a,b])$ is a bounded subset of $\mathbb{R}$. Let $m = \inf f([a,b])$, $M = \sup f([a,b])$. For any $n \in \mathbb{N}$ the number $M - \frac{1}{n}$ is not an upper bound of the set $f([a,b])$ while $m + \frac{1}{n}$ is not a lower bound of $f([a,b])$. Hence we can find points $y_n, z_n \in [a,b]$ such that $f(y_n) > M - \frac{1}{n}$ and $f(z_n) < m + \frac{1}{n}$. At the same time, $m \leq f(x) \leq M$ for all $x \in [a,b]$. It follows that $f(y_n) \to M$ and $f(z_n) \to m$ as $n \to \infty$. By the Bolzano-Weierstrass Theorem, the sequence $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ converging to some $c_1$. The sequence $\{z_n\}$ also has a subsequence $\{z_{m_k}\}$ converging to some $c_2$. Moreover, $c_1, c_2 \in [a,b]$. The continuity of $f$ implies that $f(y_{n_k}) \to f(c_1)$ and $f(z_{m_k}) \to f(c_2)$ as $k \to \infty$. Since $\{f(y_{n_k})\}$ is a subsequence of $\{f(y_n)\}$ and $\{f(z_{m_k})\}$ is a subsequence of $\{f(z_n)\}$, we conclude that $f(c_1) = M$ and $f(c_2) = m$. Thus the function $f$ attains its maximum $M$ on the interval $[a,b]$ at the point $c_1$ and its minimum $m$ at the point $c_2$.

**Problem 4.** Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(-1) = f(0) = f(1) = 0$ and $f(x) = \dfrac{x-1}{x^2-1} \sin \dfrac{1}{x}$ for $x \in \mathbb{R} \setminus \{-1, 0, 1\}$.

**(i)** Determine all points at which the function $f$ is continuous.

The polynomial functions $g_1(x) = x - 1$ and $g_2(x) = x^2 - 1$ are continuous on the entire real line. Moreover, $g_2(x) = 0$ if and only if $x = 1$ or $-1$. Therefore the quotient $g(x) = g_1(x)/g_2(x)$ is well defined and continuous on $\mathbb{R} \setminus \{-1, 1\}$.

Further, the function $h_1(x) = 1/x$ is continuous on $\mathbb{R} \setminus \{0\}$. Since the function $h_2(x) = \sin x$ is continuous on $\mathbb{R}$, the composition function $h(x) = h_2(h_1(x))$ is continuous on $\mathbb{R} \setminus \{0\}$.

Clearly, $f(x) = g(x)h(x)$ for all $x \in \mathbb{R} \setminus \{-1, 0, 1\}$. It follows that the function $f$ is continuous on $\mathbb{R} \setminus \{-1, 0, 1\}$.

It remains to determine whether the function $f$ is continuous at points $-1$, 0, and 1. Observe that $g(x) = 1/(x+1)$ for all $x \in \mathbb{R} \setminus \{-1, 1\}$. Therefore $g(x) \to 1$ as $x \to 0$, $g(x) \to 1/2$ as $x \to 1$, and $g(x) \to \pm\infty$ as $x \to -1$. Since the function $h$ is continuous at $-1$ and 1, we have $h(x) \to h(-1) = -\sin 1$ as $x \to -1$ and $h(x) \to h(1) = \sin 1$ as $x \to 1$. Note that $\sin 1 \neq 0$ since $0 < 1 < \pi/2$. It follows that $f(x) \to \pm\infty$ as $x \to -1$. In particular, $f$ is discontinuous at $-1$.

Further, $f(x) \to \frac{1}{2}\sin 1$ as $x \to 1$. Since $f(1) = 0$, the function $f$ has a removable discontinuity at 1.

Finally, the function $f$ is not continuous at 0 since it has no limit at 0. To be precise, let $x_n = (\pi/2 + 2\pi n)^{-1}$ and $y_n = (-\pi/2 + 2\pi n)^{-1}$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ and $\{y_n\}$ are two sequences of positive numbers converging to 0. We have $h(x_n) = 1$ and $h(y_n) = -1$ for all $n \in \mathbb{N}$. It follows that $f(x_n) \to 1$ and $f(y_n) \to -1$ as $n \to \infty$. Hence there is no limit of $f(x)$ as $x \to 0+$.

**(ii)** Is the function $f$ uniformly continuous on the interval $(0,1)$? Is it uniformly continuous on the interval $(1,2)$?

Any function uniformly continuous on the open interval $(0,1)$ can be extended to a continuous function on $[0,1]$. As a consequence, such a function has a right-hand limit at 0. However we already know that the function $f$ has no right-hand limit at 0. Therefore $f$ is not uniformly continuous on $(0,1)$.

The function $f$ is continuous on $(1,2]$ and has a removable singularity at 1. Changing the value of $f$ at 1 to the limit at 1, we obtain a function continuous on $[1,2]$. It is known that every function continuous on the closed interval $[1,2]$ is also uniformly continuous on $[1,2]$. Further, any function uniformly continuous on the set $[1,2]$ is also uniformly continuous on its subset $(1,2)$. Since the redefined function coincides with $f$ on $(1,2)$, we conclude that $f$ is uniformly continuous on $(1,2)$.

**Bonus Problem 5.** Given a set $X$, let $\mathcal{P}(X)$ denote the set of all subsets of $X$. Prove that $\mathcal{P}(X)$ is not of the same cardinality as $X$.

*Proof:* We have to prove that there is no bijective map of $X$ onto $\mathcal{P}(X)$. Let us consider an arbitrary map $f : X \to \mathcal{P}(X)$. The image $f(x)$ of an element $x \in X$ under this map is a subset of $X$. We define a set

$$E = \{x \in X \mid x \notin f(x)\}.$$

By definition of the set $E$, any element $x \in X$ belongs to $E$ if and only if it does not belong to $f(x)$. As a consequence, $E \neq f(x)$ for all $x \in X$. Hence the map $f$ is not onto. In particular, it is not bijective.