MATH 409

Advanced Calculus I

Lecture 15: Derivatives of elementary functions. Derivative of the inverse function.

The derivative

Definition. A real function f is said to be **differentiable** at a point $a \in \mathbb{R}$ if it is defined on an open interval containing a and the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The limit is denoted $f'(a)$ and called the **derivative** of f at a.

An equivalent condition is $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

Remark. The one-sided limits $\lim_{x \to a+} \frac{f(x)-f(a)}{x-a}$ and $\lim_{x \to a-} \frac{f(x)-f(a)}{x-a}$ are called the right-hand and left-hand derivatives of f at a. One of them or both might exist even if f is not differentiable at a.

Differentiability theorems

Theorem If functions f and g are differentiable at a point $a \in \mathbb{R}$, then their sum $f + g$, difference $f - g$, and product $f \cdot g$ are also differentiable at a. Moreover,

$$(f+g)'(a) = f'(a) + g'(a),$$
$$(f-g)'(a) = f'(a) - g'(a),$$
$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

If, additionally, $g(a) \neq 0$ then the quotient f/g is also differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Theorem If a function f is differentiable at a point $a \in \mathbb{R}$ and a function g is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a. Moreover,

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

The derivative as a function

Definition. A function f is said to be **differentiable** on an open interval (c, d) if it is differentiable at each point of (c, d). It is said to be **differentiable** on a closed interval $[c, d]$ if it is differentiable on the open interval (c, d) and, additionally, there exist the right-hand derivative of f at c and the left-hand derivative at d.

Suppose that a function f is differentiable on an interval I. Then the derivative of f can be regarded as a function on I.

Notation: f'. *Alternative notation:* $\dot{f}$, $\dfrac{df}{dx}$, $D_x f$, $f^{(1)}$.

The value of the derivative function at a point $a \in I$ is denoted $f'(a)$ or $(f(x))'|_{x=a}$.
For example, the derivative of a function $f(x) = x^2$ at 2 can be denoted $f'(2)$ or $(x^2)'|_{x=2}$, but not $(2^2)'$.

Higher-order derivatives

Higher-order derivatives of a function f are defined inductively. Namely, for any integer $n \geq 2$ and any $a \in \mathbb{R}$, the ***n*-th derivative** of f at the point a, denoted $f^{(n)}(a)$, is defined by $f^{(n)}(a) = (f^{(n-1)})'(a)$.

Let I be an interval of the real line $\mathbb{R}$. We denote by $C(I)$ or $C^0(I)$ the set of all continuous functions on I. For any $n \in \mathbb{N}$ we denote by $C^n(I)$ the set of all functions $f : I \to \mathbb{R}$ that are n **times continuously differentiable** on I, i.e., the n-th derivative $f^{(n)}$ is well-defined and continuous on I. Finally, $C^\infty(I)$ denotes the set of all functions $f : I \to \mathbb{R}$ that are **infinitely differentiable** on I, i.e., $f^{(n)}(a)$ is well-defined for all $n \in \mathbb{N}$ and $a \in I$.

We know that every function differentiable at a point a is also continuous at a. It follows that $C^{n+1}(I) \subset C^n(I)$ for all $n \geq 0$. Besides, $C^\infty(I) = \bigcap_{n \geq 0} C^n(I)$.

Examples of differentiable functions

- $1' = 0$.
- $x' = 1$.
- $(x^2)' = 2x$.
- $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ on $\mathbb{R} \setminus \{0\}$.
- $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$ on $(0, \infty)$.
- $(\sin x)' = \cos x$.
- $(\cos x)' = -\sin x$.
- $(\tan x)' = \frac{1}{\cos^2 x}$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Examples

- $f(0) = 0, \;\; f(x) = x \sin \frac{1}{x}, \;\; x \neq 0.$

Using the Product Rule and the Chain Rule, we obtain that the function f is differentiable on $\mathbb{R} \setminus \{0\}$. Moreover, for any $x \neq 0$,

$$\begin{aligned} f'(x) = \left(x \sin \frac{1}{x}\right)' &= \sin \frac{1}{x} + x \left(\sin \frac{1}{x}\right)' \\ &= \sin \frac{1}{x} + x \sin' \frac{1}{x} \left(\frac{1}{x}\right)' = \sin \frac{1}{x} + x \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) \\ &= \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}. \end{aligned}$$

Also, we know that f is continuous at 0. However it is not differentiable at 0. Indeed, $\dfrac{f(h) - f(0)}{h} = \sin \dfrac{1}{h}$, which has no limit as $h \to 0$.

Examples

- $g(0) = 0, \;\; g(x) = x^2 \sin \frac{1}{x}, \;\; x \neq 0.$

Using the Product Rule and the previous example, we obtain that the function g is differentiable on $\mathbb{R} \setminus \{0\}$. Moreover, for any $x \neq 0$,

$$g'(x) = \left(x \cdot x \sin \frac{1}{x} \right)' = x \sin \frac{1}{x} + x \left(x \sin \frac{1}{x} \right)'$$
$$= x \sin \frac{1}{x} + x \left(\sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

The function g is differentiable at 0 as well. Indeed,

$$\frac{g(h) - g(0)}{h} = h \sin \frac{1}{h} \to 0 \;\; \text{as} \;\; h \to 0.$$

Notice that g is not continuously differentiable on $\mathbb{R}$ since g' is not continuous at 0. Namely, $\lim\limits_{x \to 0} g'(x)$ does not exist.

Power rule: integer exponents

Theorem $(x^n)' = nx^{n-1}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof: The proof is by induction on n. In the case $n = 1$, we have $(x^1)' = x' = 1 = 1 \cdot x^0$ for all $x \in \mathbb{R}$. Now assume that $(x^n)' = nx^{n-1}$ for some $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. Using the Product Rule, we obtain $(x^{n+1})' = (x^n x)' = (x^n)'x + x^n x'$ $= nx^{n-1}x + x^n = (n+1)x^n$.

Remark. The theorem can also be proved directly using the formula $\dfrac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1}$.

Theorem $(x^{-n})' = -nx^{-n-1}$ for all $x \neq 0$, $n \in \mathbb{N}$.

Proof: Using the Reciprocal Rule, we obtain $(x^{-n})' = (1/x^n)' = -(x^n)'/(x^n)^2 = -nx^{n-1}/x^{2n} = -nx^{-n-1}$.

Derivative of the inverse function

Theorem Suppose f is an invertible continuous function. If f is differentiable at a point a and $f'(a) \neq 0$, then the inverse function is differentiable at the point $b = f(a)$ and, moreover,

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

Remark. In the case $f'(a) = 0$, the inverse function f^{-1} is not differentiable at $f(a)$. Indeed, if f^{-1} is differentiable at $b = f(a)$, then the Chain Rule implies that

$$(f^{-1} \circ f)'(a) = (f^{-1})'(b) \cdot f'(a).$$

Obviously, $f^{-1} \circ f$ is the identity function. Therefore $(f^{-1} \circ f)'(a) = 1 \neq 0$ so that $f'(a) \neq 0$.

Proof of the theorem: The function f is defined on an open interval $I = (c, d)$ containing a. Since f is continuous and invertible, it follows from the Intermediate Value theorem that f is strictly monotone on I, the image $f(I)$ is an open interval containing b, and the inverse function f^{-1} is continuous on $f(I)$. Besides, f^{-1} is strictly monotone on $f(I)$.

We have $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a} = f'(a)$. Since $f'(a) \neq 0$, it follows that $\lim\limits_{x\to a} \dfrac{x - a}{f(x) - f(a)} = \dfrac{1}{f'(a)}$. Since f^{-1} is continuous and monotone on the interval $f(I)$, we obtain that $f^{-1}(y) \to a$ and $f^{-1}(y) \neq a$ when $y \to b$ and $y \neq b$.

Therefore $\lim\limits_{y\to b} \dfrac{f^{-1}(y) - a}{y - b} = \lim\limits_{y\to b} \dfrac{f^{-1}(y) - a}{f(f^{-1}(y)) - b}$

$= \lim\limits_{x\to a} \dfrac{x - a}{f(x) - f(a)} = \dfrac{1}{f'(a)}.$

Example

- $f(x) = \arccos x, \quad x \in [-1, 1].$

The function $g(y) = \cos y$ is strictly decreasing on the interval $[0, \pi]$ and maps this interval onto $[-1, 1]$. By definition, the function $f(x) = \arccos x$ is the inverse of the restriction of g to $[0, \pi]$. Notice that $g'(0) = g'(\pi) = 0$ and $g'(y) \neq 0$ for $y \in (0, \pi)$. It follows that the function f is differentiable on $(-1, 1)$ and not differentiable at 1 and -1. Moreover, for any $x \in (-1, 1)$,

$$f'(x) = \frac{1}{g'(f(x))} = -\frac{1}{\sin(\arccos x)}.$$

Let $y = \arccos x$. We have $\sin^2 y + \cos^2 y = 1$. Besides, $\sin y > 0$ since $y \in (0, \pi)$. Consequently, $\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$. Thus $f'(x) = -\dfrac{1}{\sqrt{1 - x^2}}$.

Exponential and logarithmic functions

Theorem The sequence $x_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$ is increasing and bounded, hence convergent.

The limit is the number $e = 2.718281828\ldots$ (*"I'm forming a mnemonic to remember a constant in analysis"*).

Corollary $\lim\limits_{x \to 0} (1 + x)^{1/x} = e$.

For any $a > 0$, $a \neq 1$ the exponential function $f(x) = a^x$ is strictly monotone and continuous on $\mathbb{R}$. It maps $\mathbb{R}$ onto $(0, \infty)$. Therefore the inverse function $g(y) = \log_a y$ is strictly monotone and continuous on $(0, \infty)$. The natural logarithm $\log_e y$ is also denoted $\log y$.

Since $(1 + h)^{1/h} \to e$ as $h \to 0$, it follows that $h^{-1} \log(1 + h) = \log(1 + h)^{1/h} \to \log e = 1$ as $h \to 0$. In other words, $(\log y)'|_{y=1} = 1$. This implies that $(e^x)'|_{x=0} = 1$.

Examples

- $f(x) = e^x, \;\; x \in \mathbb{R}.$

$$\frac{f(x+h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h} = \frac{e^x e^h - e^x}{h} = \frac{e^x(e^h - 1)}{h}$$

for all $x, h \in \mathbb{R}$. Therefore for any $x \in \mathbb{R}$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x f'(0) = e^x.$$

- $f(x) = a^x, \;\; x \in \mathbb{R}, \;\;$ where $a > 0.$

$f(x) = e^{\log a^x} = e^{x \log a}$ so that $f'(x) = e^{x \log a} \log a = a^x \log a.$

- $f(x) = \log x, \;\; x \in (0, \infty).$

Since f is the inverse of the function $g(y) = e^y$, we obtain $f'(x) = 1/g'(\log x) = 1/e^{\log x} = 1/x$ for all $x > 0$.

Power rule: general case

Theorem $(x^\alpha)' = \alpha x^{\alpha-1}$ for all $x > 0$ and $\alpha \in \mathbb{R}$.

Proof: Let us fix a number $\alpha \in \mathbb{R}$ and consider a function $f(x) = x^\alpha$, $x \in (0, \infty)$. For any $x > 0$ we obtain $f(x) = e^{\log(x^\alpha)} = e^{\alpha \log x} = a^{\log x}$, where $a = e^\alpha$. Hence $f = h \circ g$, where $g(x) = \log x$, $x > 0$ and $h(y) = a^y$, $y \in \mathbb{R}$. By the Chain Rule,

$$
\begin{aligned}
f'(x) = h'(g(x)) \cdot g'(x) &= a^{\log x} \log a \cdot (\log x)' \\
&= f(x) \log a \cdot (\log x)' = f(x) \cdot \alpha (\log x)' \\
&= f(x) \cdot \alpha / x = x^\alpha \cdot \alpha / x = \alpha x^{\alpha-1}.
\end{aligned}
$$