

MATH 409

Advanced Calculus I

Lecture 16:

Mean value theorem.

Taylor's formula.

Points of local extremum

Let $f : E \rightarrow \mathbb{R}$ be a function defined on a set $E \subset \mathbb{R}$.

Definition. We say that f attains a **local maximum** at a point $c \in E$ if there exists $\varepsilon > 0$ such that $f(x) \leq f(c)$ for all $x \in E \cap (c - \varepsilon, c + \varepsilon)$. Similarly, f attains a **local minimum** at $c \in E$ if there exists $\varepsilon > 0$ such that $f(x) \geq f(c)$ for all $x \in E \cap (c - \varepsilon, c + \varepsilon)$.

Theorem (Fermat) If a function f is differentiable at a point c of local extremum (maximum or minimum), then $f'(c) = 0$.

Proof: Assume c is a point of local minimum. Since f is defined on an open interval containing c , there exists $\varepsilon > 0$ such that $f(c+h) - f(c) \geq 0$ for all h , $|h| < \varepsilon$. In the case $h > 0$ this implies $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h} \geq 0$. In the case $h < 0$ this implies $\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h} \leq 0$. Since both limits are equal to $f'(c)$, we conclude that $f'(c) = 0$.

Rolle's Theorem

Theorem Suppose that $a, b \in \mathbb{R}$ with $a < b$. If a function f is continuous on the interval $[a, b]$, differentiable on (a, b) , and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Proof: By the Extreme Value Theorem, the function f attains its (absolute) maximum M and minimum m on $[a, b]$. In the case $M \neq m$, at least one of the extrema is attained at a point $c \in (a, b)$. Then $f'(c) = 0$ due to Fermat's theorem. In the case $M = m$, the function f is constant on $[a, b]$. Then $f'(c) = 0$ for all $c \in (a, b)$.

Corollary If a polynomial $P(x)$ has $k > 1$ distinct real roots, then the polynomial $P'(x)$ has at least $k-1$ distinct real roots.

Proof: Let x_1, x_2, \dots, x_k be distinct real roots of $P(x)$ ordered so that $x_1 < x_2 < \dots < x_k$. By Rolle's Theorem, the derivative $P'(x)$ has a root in each of $k-1$ intervals $(x_1, x_2), (x_2, x_3), \dots, (x_{k-1}, x_k)$.

Intermediate Value Theorem for derivatives

Theorem Suppose that a function f is differentiable on an interval $[a, b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number that lies between $f'(a)$ and $f'(b)$, then $f'(x_0) = y_0$ for some $x_0 \in (a, b)$.

Remark. Although the function f is differentiable on $[a, b]$, the derivative f' need not be continuous on $[a, b]$. Hence the Intermediate Value Theorem for continuous functions may not apply here.

Corollary If a function f is differentiable on an open interval (c, d) , then the derivative f' has no jump discontinuities in (c, d) .

Proof of the theorem: First we consider the case when $f'(a) < 0$, $f'(b) > 0$, and $y_0 = 0$. Since f is differentiable on $[a, b]$, it is continuous on $[a, b]$. By the Extreme Value Theorem, f attains its absolute minimum on $[a, b]$ at some point x_0 . Since $f'(a) < 0$, we have $f(a+h) - f(a) < 0$ for $h > 0$ sufficiently small. Hence $x_0 \neq a$. Similarly, $f'(b) > 0$ implies that $f(b+h) - f(b) < 0$ for $h < 0$ sufficiently small. Hence $x_0 \neq b$. We obtain that $x_0 \in (a, b)$. Then $f'(x_0) = 0$ due to Fermat's theorem.

Next we consider the case when $f'(a) > 0$, $f'(b) < 0$, and $y_0 = 0$. Then the function $g = -f$ is differentiable on $[a, b]$ with $g'(a) = -f'(a) < 0$ and $g'(b) = -f'(b) > 0$. By the above, $g'(x_0) = 0$ for some $x_0 \in (a, b)$. Then $f'(x_0) = 0$.

In the general case, we consider a function $h(x) = f(x) - y_0x$. It is differentiable on $[a, b]$ and $h'(x) = f'(x) - y_0$ for all $x \in [a, b]$. It follows that 0 lies between $h'(a)$ and $h'(b)$. By the above, $h'(x_0) = 0$ for some $x_0 \in (a, b)$. Then $f'(x_0) = h'(x_0) + y_0 = y_0$.

Mean Value Theorem

Theorem If a function f is continuous on $[a, b]$ and differentiable on (a, b) , then there is $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Proof: Let $h_0(x) = \frac{f(a)(b-x) + f(b)(x-a)}{b-a}$, $x \in \mathbb{R}$.

We observe that the function h_0 is differentiable. Moreover, $h'_0(x) = \frac{f(b)-f(a)}{b-a}$ for all $x \in \mathbb{R}$. By construction, $h_0(a) = f(a)$ and $h_0(b) = f(b)$. It follows that the function $h = f - h_0$ is continuous on $[a, b]$, differentiable on (a, b) , and satisfies $h(a) = h(b) = 0$. By Rolle's Theorem, $h'(c) = 0$ for some $c \in (a, b)$. We have

$$h'(c) = f'(c) - h'_0(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Thus $f'(c) = (f(b) - f(a))/(b - a)$ or, equivalently, $f(b) - f(a) = f'(c)(b - a)$.

Monotone functions (revisited)

Theorem Suppose that a function f is continuous on an interval $[a, b]$ and differentiable on (a, b) .

- (i) f is increasing on $[a, b]$ if and only if $f' \geq 0$ on (a, b) .
- (ii) f is decreasing on $[a, b]$ if and only if $f' \leq 0$ on (a, b) .
- (iii) If $f' > 0$ on (a, b) , then f is strictly increasing on $[a, b]$.
- (iv) If $f' < 0$ on (a, b) , then f is strictly decreasing on $[a, b]$.
- (v) f is constant on $[a, b]$ if and only if $f' = 0$ on (a, b) .

Proof: Let $a \leq x_1 < x_2 \leq b$. By the Mean Value Theorem, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some $c \in (x_1, x_2)$. Obviously, $f'(c) > 0$ if and only if $f(x_1) < f(x_2)$. Likewise, $f'(c) \geq 0$ if and only if $f(x_1) \leq f(x_2)$. This proves statements (iii), (iv), and the “if” part of statements (i), (ii). The “only if” part of statements (i) and (ii) follows from the Comparison Theorem. Finally, statement (v) follows from statements (i) and (ii).

Generalized Mean Value Theorem

Theorem If function f and g are continuous on $[a, b]$ and differentiable on (a, b) , then there exists $c \in (a, b)$ such that

$$g'(c) (f(b) - f(a)) = f'(c) (g(b) - g(a)).$$

Proof: For any $x \in [a, b]$, let

$$h(x) = f(x) (g(b) - g(a)) - g(x) (f(b) - f(a)).$$

We observe that the function h is continuous on $[a, b]$ and differentiable on (a, b) . Further,

$$\begin{aligned} h(a) &= f(a) (g(b) - g(a)) - g(a) (f(b) - f(a)) \\ &= f(a) g(b) - g(a) f(b), \end{aligned}$$

$$\begin{aligned} h(b) &= f(b) (g(b) - g(a)) - g(b) (f(b) - f(a)) \\ &= -f(b) g(a) + g(b) f(a). \end{aligned}$$

Hence $h(a) = h(b)$. By Rolle's Theorem, $h'(c) = 0$ for some $c \in (a, b)$. It remains to notice that

$$h'(c) = f'(c) (g(b) - g(a)) - g'(c) (f(b) - f(a)).$$

Taylor's formula

Theorem Let $n \in \mathbb{N}$ and $I \subset \mathbb{R}$ be an open interval. If a function $f : I \rightarrow \mathbb{R}$ is $n + 1$ times differentiable on I , then for each pair of points $x, x_0 \in I$ there is a point c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{k=1}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}.$$

Remark. The function

$$P_n^{f, x_0}(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

is a polynomial of degree at most n . It is called the **Taylor polynomial** of order n generated by f centered at x_0 . One can check that $P_n^{f, x_0}(x_0) = f(x_0)$ and $(P_n^{f, x_0})^{(k)}(x_0) = f^{(k)}(x_0)$ for $1 \leq k \leq n$.

Proof of the theorem: Let us fix $x \in I$ and define functions

$$F(t) = \frac{(x-t)^{n+1}}{(n+1)!} \quad \text{and} \quad G(t) = f(x) - f(t) - \sum_{k=1}^n \frac{f^{(k)}(t)}{k!} (x-t)^k.$$

The function F is infinitely differentiable on \mathbb{R} . The function G is defined and differentiable on I . By the Generalized Mean Value Theorem, for every $x_0 \in I$, $x_0 \neq x$, there exists a point c between x_0 and x such that

$$G'(c) (F(x) - F(x_0)) = F'(c) (G(x) - G(x_0)).$$

Note that the latter follows both in the case $x_0 < x$ and in the case $x < x_0$. Clearly, $F(x) = G(x) = 0$. Further,

$$\frac{d}{dt} \left(\frac{f^{(k)}(t)}{k!} (x-t)^k \right) = \frac{f^{(k+1)}(t)}{k!} (x-t)^k - \frac{f^{(k)}(t)}{(k-1)!} (x-t)^{k-1}.$$

Summing up over k from 1 to n , we obtain that

$$G'(t) = - \frac{f^{(n+1)}(t)}{n!} (x-t)^n. \quad \text{Finally, } F'(t) = - \frac{(x-t)^n}{n!} \text{ so that}$$

$G'(t)/F'(t) = f^{(n+1)}(t)$ for $t \neq x$. It follows that

$G(x_0) = f^{(n+1)}(c) F(x_0)$, which implies Taylor's formula.