MATH 409

Advanced Calculus I

Lecture 18:
Darboux sums.
The Riemann integral.

Partitions of an interval

Definition. A **partition** of a closed bounded interval $[a, b]$ is a finite subset $P \subset [a, b]$ that includes the endpoints a and b.

Let $x_0, x_1, \ldots, x_n$ be the list of all elements of P ordered so that $x_0 < x_1 < \cdots < x_n$ (note that $x_0 = a$ and $x_n = b$). These points split the interval $[a, b]$ into finitely many subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$. The **norm** of the partition P, denoted $\|P\|$, is the maximum of lengths of those subintervals: $\|P\| = \max_{1 \le j \le n} |x_j - x_{j-1}|$.

Given two partitions P and Q of the same interval, we say that Q is a **refinement** of P (or that Q is **finer** than P) if $P \subset Q$. Observe that $P \subset Q$ implies $\|Q\| \le \|P\|$.

For any two partitions P and Q of the interval $[a, b]$, the union $P \cup Q$ is also a partition that refines both P and Q.

Darboux sums

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of an interval $[a, b]$, where $x_0 = a < x_1 < \cdots < x_n = b$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

Definition. The **upper Darboux sum** (or the **upper Riemann sum**) of the function f over the partition P is the number

$$U(f, P) = \sum_{j=1}^{n} M_j(f)\, \Delta_j,$$

where $\Delta_j = x_j - x_{j-1}$ and $M_j(f) = \sup f([x_{j-1}, x_j])$ for $j = 1, 2, \dots, n$. Likewise, the **lower Darboux sum** (or the **lower Riemann sum**) of f over P is the number

$$L(f, P) = \sum_{j=1}^{n} m_j(f)\, \Delta_j,$$

where $m_j(f) = \inf f([x_{j-1}, x_j])$ for $j = 1, 2, \dots, n$.

Properties of the Darboux sums

- $L(f, P) \le U(f, P)$.

Indeed, $\inf f(J) \le \sup f(J)$ for any subinterval $J \subset [a, b]$.

- $U(f, P) \le \sup f([a, b]) \cdot (b - a)$.

We have $\sup f(J) \le \sup f([a, b])$ for any subinterval $J \subset [a, b]$. Then $\sup f(J) \cdot |J| \le \sup f([a, b]) \cdot |J|$, where $|J|$ is the length of J. Summing up over all subintervals J created by the partition P, we obtain $U(f, P) \le \sup f([a, b]) \cdot (b - a)$.

- $\inf f([a, b]) \cdot (b - a) \le L(f, P)$.

The proof is analogous to the previous one.

Remark. Observe that $\sup f([a, b]) \cdot (b - a) = U(f, P_0)$ and $\inf f([a, b]) \cdot (b - a) = L(f, P_0)$, where P_0 is the trivial partition: $P_0 = \{a, b\}$.

Properties of the Darboux sums

- $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$ for any partition Q that refines P.

Every subinterval J created by the partition P is the union of one or more subintervals $J_1, J_2, \ldots, J_k$ created by Q. Since $\sup f(J_i) \leq \sup f(J)$ for $1 \leq i \leq k$, it follows that $\sum_{i=1}^{k} \sup f(J_i) \cdot |J_i| \leq \sup f(J) \cdot \sum_{i=1}^{k} |J_i| = \sup f(J) \cdot |J|$. Summing up this inequality over all subintervals J, we obtain $U(f, Q) \leq U(f, P)$. The inequality $L(f, P) \leq L(f, Q)$ is proved in a similar way.

- $L(f, P) \leq U(f, Q)$ for any partitions P and Q.

Since the partition $P \cup Q$ refines both P and Q, it follows from the above that $L(f, P) \leq L(f, P \cup Q)$ and $U(f, P \cup Q) \leq U(f, Q)$. Besides, $L(f, P \cup Q) \leq U(f, P \cup Q)$.

Upper and lower integrals

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function.

Definition. The **upper integral** of f on $[a, b]$, denoted
$\overline{\int_a^b} f(x)\,dx$ or $(U)\int_a^b f(x)\,dx$, is the number

$$\inf \{ U(f, P) \mid P \text{ is a partition of } [a, b] \}.$$

Similarly, the **lower integral** of f on $[a, b]$, denoted
$\underline{\int_a^b} f(x)\,dx$ or $(L)\int_a^b f(x)\,dx$, is the number

$$\sup \{ L(f, P) \mid P \text{ is a partition of } [a, b] \}.$$

Remark. Since $-\infty < L(f, P) \leq U(f, Q) < +\infty$ for all partitions P and Q, it follows that

$$-\infty < (L)\int_a^b f(x)\,dx \leq (U)\int_a^b f(x)\,dx < +\infty.$$

Integrability

Definition. A bounded function $f : [a, b] \to \mathbb{R}$ is called **integrable** (or **Riemann integrable**) on the interval $[a, b]$ if the upper and lower integrals of f on $[a, b]$ coincide. The common value is called the **integral** of f on $[a, b]$ (or over $[a, b]$) and denoted $\int_a^b f(x)\,dx$.

Theorem A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

Proof of the theorem: The "if" part of the theorem follows since

$$0 \leq (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx \leq U(f,P) - L(f,P)$$

for any partition P. Conversely, assume that f is integrable on $[a,b]$. Given $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f,P) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}.$$

Also, there exists a partition Q of $[a,b]$ such that

$$L(f,Q) > \int_a^b f(x)\,dx - \frac{\varepsilon}{2}.$$

Then $U(f,P) - L(f,Q) < \varepsilon$. Now $P \cup Q$ is a partition of $[a,b]$ that refines both P and Q. It follows that $U(f,P\cup Q) \leq U(f,P)$ and $L(f,P\cup Q) \geq L(f,Q)$. Hence $U(f,P\cup Q) - L(f,P\cup Q) \leq U(f,P) - L(f,Q) < \varepsilon$.

Examples

- Constant function $f(x) = c$ is integrable on any interval $[a, b]$ and $\int_a^b f(x)\,dx = c(b-a)$.

Indeed, for the trivial partition $P_0 = \{a, b\}$ we obtain
$U(f, P_0) = c(b-a) = L(f, P_0)$.

- Step function $f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$ is integrable on $[-1, 1]$ and $\int_{-1}^{1} f(x)\,dx = 1$.

For any $\varepsilon \in (0, 1)$ consider a partition $P_\varepsilon = \{-1, -\varepsilon, \varepsilon, 1\}$.
Then $U(f, P_\varepsilon) = 1 + \varepsilon$ and $L(f, P_\varepsilon) = 1 - \varepsilon$.

Examples

- Dirichlet function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

is not integrable on any interval $[a, b]$.

Indeed, any subinterval of $[a, b]$ contains both rational and irrational points. Therefore $U(f, P) = b - a$ and $L(f, P) = 0$ for all partitions of $[a, b]$.

- Riemann function $f(x) = \begin{cases} 1/q & \text{if } x = p/q, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

is integrable on any interval $[a, b]$.

For any $\delta > 0$ the interval $[a, b]$ contains only finitely many points $y_1, y_2, \ldots, y_k$ such that $f(y_i) \geq \delta$. Let P_δ be a partition of $[a, b]$ that includes points $y_i \pm \delta/k$. Then $L(f, P_\delta) = 0$ and $U(f, P_\delta) \leq 2\delta + \delta(b - a)$.

Continuity $\implies$ integrability

Theorem If a function $f : [a, b] \to \mathbb{R}$ is continuous on the interval $[a, b]$, then it is integrable on $[a, b]$.

Proof: Since the function f is continuous, it is bounded on $[a, b]$. Furthermore, f is uniformly continuous on $[a, b]$. Therefore for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/(b - a)$ for all $x, y \in [a, b]$. Obviously, there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ that satisfies $\|P\| < \delta$. Let $J = [x_{j-1}, x_j]$ be an arbitrary subinterval of $[a, b]$ created by P. By the Extreme Value Theorem, there are points $x_-, x_+ \in J$ such that $f(x_+) = \sup f(J)$ and $f(x_-) = \inf f(J)$. Since $\|P\| < \delta$, the length of J satisfies $|J| < \delta$. Then $|x_+ - x_-| \leq |J| < \delta$ so that $|f(x_+) - f(x_-)| < \varepsilon/(b - a)$. It follows that $\sup f(J) \cdot |J| - \inf f(J) \cdot |J| < \varepsilon |J|/(b - a)$. Summing up the latter inequality over all subintervals J, we obtain that $U(f, P) - L(f, P) < \varepsilon$. Thus f is integrable.

Riemann sums

Definition. A **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ with respect to a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\,(x_j - x_{j-1}).$$

Remark. Note that the function f need not be bounded. If f is bounded, then $L(f, P) \le \mathcal{S}(f, P, t_j) \le U(f, P)$ for any choice of samples t_j.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

Theorem The Riemann sums $\mathcal{S}(f, P, t_j)$ converge to a limit $I(f)$ as $\|P\| \to 0$ if and only if the function f is integrable on $[a, b]$ and $I(f) = \int_a^b f(x)\,dx$.