MATH 409–501/503 Fall 2013

Sample problems for Test 2: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (20 pts.) Prove the Chain Rule: if a function f is differentiable at a point c and a function g is differentiable at $f(c)$, then the composition $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Since the function f is differentiable at the point c, the domain of f contains an open interval $I_0 = (c - \delta_0, c + \delta_0)$ for some $\delta_0 > 0$. Since g is differentiable at $f(c)$, the domain of g contains an open interval $J = (f(c) - \varepsilon_0, f(c) + \varepsilon_0)$ for some $\varepsilon_0 > 0$. The differentiability of f at c implies that f is continuous at that point. Hence there exists $\delta_1 \in (0, \delta_0)$ such that $|f(x) - f(c)| < \varepsilon_0$ whenever $|x - c| < \delta_1$. Then the composition $g \circ f$ is well defined on the interval $I_1 = (c - \delta_1, c + \delta_1)$. Consider a set $E = \{x \in I_1 \mid f(x) \neq f(c)\}$. For any $x \in E$,

$$\frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}.$$

As $x \to c$ within the subset E, we have $f(x) \to f(c)$ while $f(x) \neq f(c)$. Therefore

$$\lim_{\substack{x \to c \\ x \in E}} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = \lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c)).$$

Consequently,

$$\lim_{\substack{x \to c \\ x \in E}} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = g'(f(c)) \cdot f'(c).$$

In the case $f(x) \neq f(c)$ for all x in a sufficiently small punctured neighborhood of c, the restriction $x \in E$ in the above limit is redundant and we are done. Otherwise we also need to consider the limit as $x \to c$ within the complement of E. Notice that $g(f(x)) - g(f(c)) = f(x) - f(c) = 0$ for all $x \notin E$. Hence

$$\lim_{\substack{x \to c \\ x \notin E}} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = \lim_{\substack{x \to c \\ x \notin E}} \frac{f(x) - f(c)}{x - c} = 0.$$

In particular, we have $f'(c) = 0$ in this case so that

$$\lim_{\substack{x \to c \\ x \notin E}} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = g'(f(c)) \cdot f'(c).$$

Problem 2 (25 pts.) Find the following limits of functions:

(i) $\lim_{x \to 0} (1 + x)^{1/x}$, (ii) $\lim_{x \to +\infty} (1 + x)^{1/x}$, (iii) $\lim_{x \to 0+} x^x$.

The function $f(x) = (1+x)^{1/x}$ is well defined on $(-1,0) \cup (0,\infty)$. Since $f(x) > 0$ for all $x > -1$, $x \neq 0$, a function $g(x) = \log f(x)$ is well defined on $(-1,0)\cup(0,\infty)$ as well. For any $x > -1$, $x \neq 0$, we have $g(x) = \log(1+x)^{1/x} = x^{-1}\log(1+x)$. Hence $g = h_1/h_2$, where the functions $h_1(x) = \log(1+x)$ and $h_2(x) = x$ are continuously differentiable on $(-1,\infty)$. Since $h_1(0) = h_2(0) = 0$, it follows that $\lim_{x\to 0} h_1(x) = \lim_{x\to 0} h_2(x) = 0$. By l'Hôpital's Rule,

$$\lim_{x\to 0} \frac{h_1(x)}{h_2(x)} = \lim_{x\to 0} \frac{h_1'(x)}{h_2'(x)}$$

assuming the latter limit exists. Since $h_1'(0) = (1+x)^{-1}|_{x=0} = 1$ and $h_2'(0) = 1$, we obtain

$$\lim_{x\to 0} \frac{h_1(x)}{h_2(x)} = \lim_{x\to 0} \frac{h_1'(x)}{h_2'(x)} = \frac{\lim_{x\to 0} h_1'(x)}{\lim_{x\to 0} h_2'(x)} = \frac{1}{1} = 1.$$

Further, $\lim_{x\to+\infty} h_1(x) = \lim_{x\to+\infty} h_2(x) = +\infty$. At the same time, $h_1'(x) = (1+x)^{-1} \to 0$ as $x \to +\infty$ while h_2' is identically 1. Using l'Hôpital's Rule and a limit theorem, we obtain

$$\lim_{x\to+\infty} \frac{h_1(x)}{h_2(x)} = \lim_{x\to+\infty} \frac{h_1'(x)}{h_2'(x)} = \frac{\lim_{x\to+\infty} h_1'(x)}{\lim_{x\to+\infty} h_2'(x)} = \frac{0}{1} = 0.$$

Since $f = e^g$, a composition of g with a continuous function, it follows that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{g(x)} = \exp\left(\lim_{x\to 0} g(x)\right) = e^1 = e, \qquad \lim_{x\to+\infty} f(x) = \exp\left(\lim_{x\to+\infty} g(x)\right) = e^0 = 1.$$

Now let us consider a function $F(x) = x^x$, $x > 0$. Since F takes positive values, a function $G(x) = \log F(x)$ is well defined on $(0,\infty)$. We have $G(x) = \log(x^x) = x\log x$ for all $x > 0$ so that $G = H_1/H_2$, where $H_1(x) = \log x$ and $H_2(x) = x^{-1}$ are differentiable functions on $(0,\infty)$. We observe that $\lim_{x\to 0+} H_1(x) = -\infty$ and $\lim_{x\to 0+} H_2(x) = +\infty$. By l'Hôpital's Rule,

$$\lim_{x\to 0+} \frac{H_1(x)}{H_2(x)} = \lim_{x\to 0+} \frac{H_1'(x)}{H_2'(x)} = \lim_{x\to 0+} \frac{(\log x)'}{(x^{-1})'} = \lim_{x\to 0+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0+} (-x) = 0.$$

Consequently,

$$\lim_{x\to 0+} F(x) = \lim_{x\to 0+} e^{G(x)} = \exp\left(\lim_{x\to 0+} G(x)\right) = e^0 = 1.$$

Problem 3 (20 pts.) Find the limit of a sequence

$$x_n = \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}, \quad n = 1, 2, \ldots,$$

where k is a natural number.

The general element of the sequence can be represented as

$$x_n = \frac{1^k + 2^k + \cdots + n^k}{n^k} \cdot \frac{1}{n} = \left(\frac{1}{n}\right)^k \frac{1}{n} + \left(\frac{2}{n}\right)^k \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^k \frac{1}{n},$$

which shows that x_n is a Riemann sum of the function $f(x) = x^k$ on the interval $[0, 1]$ that corresponds to the partition $P_n = \{0, 1/n, 2/n, \dots, (n-1)/n, 1\}$ and samples $t_j = j/n$, $j = 1, 2, \dots, n$. The norm of the partition is $\|P_n\| = 1/n$. Since $\|P_n\| \to 0$ as $n \to \infty$ and the function f is integrable on $[0, 1]$, the Riemann sums x_n converge to the integral:

$$\lim_{n\to\infty} x_n = \int_0^1 x^k\,dx = \frac{x^{k+1}}{k+1}\Big|_{x=0}^{1} = \frac{1}{k+1}.$$

Problem 4 (25 pts.) Find indefinite integrals and evaluate definite integrals:

(i) $\displaystyle\int \frac{x^2}{1-x}\,dx$, (ii) $\displaystyle\int_0^{\pi} \sin^2(2x)\,dx$, (iii) $\displaystyle\int \log^3 x\,dx$,

(iv) $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$, (v) $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$.

To find the indefinite integral of a rational function $y(x) = x^2/(1-x)$, we expand it into the sum of a polynomial and a simple fraction:

$$\frac{x^2}{1-x} = \frac{x^2-1+1}{1-x} = \frac{x^2-1}{1-x} + \frac{1}{1-x} = \frac{(x-1)(x+1)}{1-x} + \frac{1}{1-x} = -x - 1 - \frac{1}{x-1}.$$

Since the domain of the function y is $(-\infty, 1) \cup (1, \infty)$, the indefinite integral has different representations on the intervals $(-\infty, 1)$ and $(1, \infty)$:

$$\int \frac{x^2}{1-x}\,dx = \int \left(-x - 1 - \frac{1}{x-1}\right) dx = \begin{cases} -x^2/2 - x - \log(1-x) + C_1, & x < 1, \\ -x^2/2 - x - \log(x-1) + C_2, & x > 1. \end{cases}$$

To integrate the function $y(x) = \sin^2(2x)$, we use a trigonometric formula $1 - \cos(2\alpha) = 2\sin^2\alpha$ and a new variable $u = 4x$:

$$\int_0^{\pi} \sin^2(2x)\,dx = \int_0^{\pi} \frac{1-\cos(4x)}{2}\,dx = \int_0^{\pi} \frac{1-\cos(4x)}{8}\,d(4x)$$

$$= \int_0^{4\pi} \frac{1-\cos u}{8}\,du = \frac{u - \sin u}{8}\Big|_{u=0}^{4\pi} = \frac{\pi}{2}.$$

To find the antiderivative of the function $y(x) = \log^3 x$, we integrate by parts three times:

$$\int \log^3 x\,dx = x\log^3 x - \int x\,d(\log^3 x) = x\log^3 x - \int x(\log^3 x)'\,dx = x\log^3 x - \int 3\log^2 x\,dx$$

$$= x\log^3 x - 3x\log^2 x + \int x\,d(3\log^2 x) = x\log^3 x - 3x\log^2 x + \int x(3\log^2 x)'\,dx$$

$$= x\log^3 x - 3x\log^2 x + \int 6\log x\,dx = x\log^3 x - 3x\log^2 x + 6x\log x - \int x\,d(6\log x)$$

$$= x\log^3 x - 3x\log^2 x + 6x\log x - \int x(6\log x)'\,dx = x\log^3 x - 3x\log^2 x + 6x\log x - \int 6\,dx$$

$$= x\log^3 x - 3x\log^2 x + 6x\log x - 6x + C.$$

To integrate the function $y(x) = x/\sqrt{1-x^2}$, we use a new variable $u = 1 - x^2$:

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_0^{1/2} \frac{(1-x^2)'}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,d(1-x^2)$$

$$= -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du = \int_{3/4}^{1} \frac{1}{2\sqrt{u}}\,du = \sqrt{u}\,\Big|_{u=3/4}^{1} = 1 - \frac{\sqrt{3}}{2}.$$

To integrate the function $y(x) = 1/\sqrt{4-x^2}$, we use a substitution $x = 2\sin t$ (observe that x changes from 0 to 1 when t changes from 0 to $\pi/6$):

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \int_0^{\pi/6} \frac{1}{\sqrt{4-(2\sin t)^2}}\,d(2\sin t) = \int_0^{\pi/6} \frac{(2\sin t)'}{\sqrt{4-4\sin^2 t}}\,dt$$

$$= \int_0^{\pi/6} \frac{2\cos t}{\sqrt{4\cos^2 t}}\,dt = \int_0^{\pi/6} \frac{2\cos t}{2\cos t}\,dt = \int_0^{\pi/6} 1\,dx = \frac{\pi}{6}.$$

Bonus Problem 5 (15 pts.) Suppose that a function $p : \mathbb{R} \to \mathbb{R}$ is locally a polynomial, which means that for every $c \in \mathbb{R}$ there exists $\varepsilon > 0$ such that p coincides with a polynomial on the interval $(c-\varepsilon, c+\varepsilon)$. Prove that p is a polynomial.

For any $c \in \mathbb{R}$ let p_c denote a polynomial and ε_c denote a positive number such that $p(x) = p_c(x)$ for all $x \in (c-\varepsilon_c, c+\varepsilon_c)$. We are going to show that $p = p_0$ on the entire real line. Consider two sets $E_+ = \{x > 0 \mid p(x) \neq p_0(x)\}$ and $E_- = \{x < 0 \mid p(x) \neq p_0(x)\}$. Assume that the set E_+ is not empty. Clearly, E_+ is bounded below. Hence $d = \inf E_+$ is a well-defined real number. Note that $E_+ \subset [\varepsilon_0, \infty)$. Therefore $d \geq \varepsilon_0 > 0$. Observe that $p(x) = p_0(x)$ for $x \in (0, d)$ and $p(x) = p_d(x)$ for $x \in (d-\varepsilon_d, d+\varepsilon_d)$. The interval $(0, d)$ overlaps with the interval $(d-\varepsilon_d, d+\varepsilon_d)$. Hence p_d coincides with p_0 on the intersection $(0, d) \cap (d-\varepsilon_d, d+\varepsilon_d)$. Equivalently, the difference $p_d - p_0$ is zero on $(0, d) \cap (d-\varepsilon_d, d+\varepsilon_d)$. Since $p_d - p_0$ is a polynomial and any nonzero polynomial has only finitely many roots, we conclude that $p_d - p_0$ is identically 0. Then the polynomials p_d and p_0 are the same. It follows that $p(x) = p_0(x)$ for $x \in (0, d+\varepsilon_d)$ so that $d \neq \inf E_+$, a contradiction. Thus $E_+ = \emptyset$. Similarly, we prove that the set E_- is empty as well. Since $E_+ = E_- = \emptyset$, the function p coincides with the polynomial p_0.

Bonus Problem 6 (15 pts.) Show that a function

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{1-x^2}\right) & \text{if } |x| < 1, \\ 0 \text{ if } |x| \geq 1 \end{cases}$$

is infinitely differentiable on $\mathbb{R}$.

Consider a function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = e^{-1/x}$ for $x > 0$ and $h(x) = 0$ for $x \leq 0$. It is easy to verify that $f(x) = h(2+2x)h(2-2x)$ for all $x \in \mathbb{R}$. Since the set $C^\infty(\mathbb{R})$ of infinitely differentiable functions is closed under multiplication and composition of functions, it is enough to prove that $h \in C^\infty(\mathbb{R})$.

Obviously, the function h is infinitely differentiable on $(-\infty, 0)$ and on $(0, \infty)$. Moreover, all derivatives on $(-\infty, 0)$ are identically zero. Let us prove that for any integer $n \geq 0$ there exists a polynomial p_n such that $h^{(n)}(x) = p_n(x)x^{-2n}e^{-1/x}$ for all $x > 0$, where $h^{(n)}$ is the n-th derivative of h for $n > 0$ and $h^{(0)} = h$. The proof is by induction on n. The base case $n = 0$ is trivial, with $p_0 = 1$. Now assume that the above representation holds for some integer $n \geq 0$. Then

$$\begin{aligned} h^{(n+1)}(x) &= \big(h^{(n)}(x)\big)' = \big(p_n(x)x^{-2n}e^{-1/x}\big)' \\ &= p_n'(x)x^{-2n}e^{-1/x} + p_n(x)(x^{-2n})'e^{-1/x} + p_n(x)x^{-2n}(e^{-1/x})' \\ &= p_n'(x)x^{-2n}e^{-1/x} + p_n(x)(-2n)x^{-2n-1}e^{-1/x} + p_n(x)x^{-2n}e^{-1/x}x^{-2} \\ &= p_{n+1}(x)x^{-2(n+1)}e^{-1/x}, \end{aligned}$$

where $p_{n+1}(x) = p_n'(x)x^2 - 2np_n(x)x + p_n(x)$ is a polynomial. Since $t^k/e^t \to 0$ as $t \to +\infty$ for any $k > 0$ and $1/x \to +\infty$ as $x \to 0+$, we obtain that $x^{-k}e^{-1/x} \to 0$ as $x \to 0+$ for any $k > 0$. Then it follows from the above that $h^{(n)}(x)/x \to 0$ as $x \to 0+$ for any $n \geq 0$. Now it easily follows by induction on $n \in \mathbb{N}$ that the function h is n times differentiable at 0 and $h^{(n)}(0) = 0$. Thus $h \in C^\infty(\mathbb{R})$.