Challenges

Challenge #1 (10 pts.)

Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Suppose that for any point $x \in \mathbb{R}$ there exists a derivative of f that vanishes at x:

$$f^{(n)}(x) = 0 \text{ for some } n \geq 1.$$

Prove that f is a polynomial.

Remark. A polynomial can be uniquely characterized as an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{(n)}(x) \equiv 0$ (identically zero) for some $n \geq 1$.

Challenges

Challenge #2 (1 pt., expired on January 24)
Construct a strict linear order $\prec$ on the set $\mathbb{C}$ of complex numbers that satisfies Axiom OA:
$a \prec b$ implies $a + c \prec b + c$ for all $a, b, c \in \mathbb{C}$.

(**+1 pt.** if you construct infinitely many different orders)

Challenge #3 (2 pts., expired on January 24)
Construct a strict linear order $\prec$ on the set $\mathbb{R}(x)$ of rational functions in variable x with real coefficients that makes $\mathbb{R}(x)$ into an ordered field.

(**+1 pt.** if you construct two different orders)

Challenges

A set $E \subset \mathbb{R}$ is called an interval if with any two elements it contains all elements of $\mathbb{R}$ that lie between them. To be precise, $a, b \in E$ and $a < c < b$ imply $c \in E$ for all $a, b, c \in \mathbb{R}$.

Challenge #4 (2 pts.)
Prove the following statements.

(i) If E is a bounded interval that consists of more than one point, then there exist $a, b \in \mathbb{R}$, $a < b$, such that $E = (a, b)$ or $[a, b)$ or $(a, b]$ or $[a, b]$.

(ii) If E is an interval that is neither bounded above nor bounded below, then $E = \mathbb{R}$.

Challenges

Consider the field $\mathbb{R}(x)$ of rational functions with the strict linear order defined by $f \prec g$ if $f(x) < g(x)$ for all $x > M$, where M is a constant depending on f and g.

A set $E \subset \mathbb{R}(x)$ is called an interval if with any two functions it contains all functions in $\mathbb{R}(x)$ that lie between them. To be precise, $f, g \in E$ and $f \prec h \prec g$ imply $h \in E$ for all $f, g, h \in \mathbb{R}(x)$.

Challenge #5 (2 pts.)
Find a nonempty bounded interval in $\mathbb{R}(x)$ that is not of the form (a, b), $[a, b)$, $(a, b]$ or $[a, b]$.

Challenges

Challenge #6 (2 pts.)
Prove that the set $\mathbb{R} \times \mathbb{R}$ is of the same cardinality as $\mathbb{R}$.

Challenge #7 (3 pts.)
Let $\mathcal{P}(\mathbb{N})$ denote the set of all subsets of $\mathbb{N}$. Prove that $\mathcal{P}(\mathbb{N})$ is of the same cardinality as $\mathbb{R}$.

Challenge #8
Let $\mathcal{F}(\mathbb{R})$ denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ and $C(\mathbb{R})$ denote the subset of continuous functions.

(5 pts.) Prove that $C(\mathbb{R})$ is of the same cardinality as $\mathbb{R}$.

(2 pts.) Prove that $\mathcal{F}(\mathbb{R})$ is not of the same cardinality as $\mathbb{R}$.

Challenges

Challenge #9

All problems are from Thomson/Bruckner/Bruckner.

Section 2.14: 2.14.14, 2.14.15, 2.14.18, 2.14.19

Each problem is worth **2 pts.**

Challenges

Challenge #10

All problems are from Thomson/Bruckner/Bruckner.

Section 2.11: 2.11.12
Section 2.13: 2.13.16, 2.13.17

Each problem is worth **3 pts.**

Challenge #11

All problems are from Thomson/Bruckner/Bruckner.

Section 3.12: 3.12.2 **(2 pts.)**, 3.12.9 **(5 pts.)**,
3.12.15 **(3 pts.)**

Challenges

Challenge #12

(4 pts.) Suppose $E \subset \mathbb{R}$ is a nonempty closed set. Prove that there exists a finite or countable subset $S \subset E$ such that $E = \overline{S}$ (that is, S is dense in E).

(3 pts.) Given a sequence $\{x_n\}$ of real numbers, let E be the set of all its limit points (i.e., limits of convergent subsequences). Prove that E is a closed set.

(4 pts.) Suppose $E \subset \mathbb{R}$ is a closed set. Prove that there exists a sequence $\{x_n\}$ such that the set of all its limit points is E.

Challenges

Challenge #13

(2 pts.) Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at all integer points and discontinuous at all non-integer points.

(5 pts.) Given a function $f : \mathbb{R} \to \mathbb{R}$, let S be the set of all points at which f is continuous. Prove that S is a countable intersection of open sets.

(5 pts.) Prove that there exists no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at all rational points and discontinuous at all irrational points.

(Hint: show that $\mathbb{Q}$ is not a countable intersection of open sets.)

Challenges

Challenge #14

(3 pts.) Prove that the Riemann function

$$R(x) = \begin{cases} 1/q & \text{if } x = p/q, \text{ a reduced fraction, } q > 0, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not differentiable at any point.

(5 pts.) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x) > 0$ for all $x \in \mathbb{Q}$ and $f(x) = 0$ for all $x \notin \mathbb{Q}$. Prove that f can not be differentiable at every irrational point.

(10 pts.) Let $R : \mathbb{R} \to \mathbb{R}$ be the Riemann function. Prove that the function R^3 is differentiable at some irrational points.

Challenges

Challenge #15

(5 pts.) Determine all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'\left(\frac{x+y}{2}\right) = \frac{f(x) - f(y)}{x - y}$ whenever $x \neq y$. Prove your answer.

(10 pts.) Construct an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$, $f(x) = 1$ if $|x| \leq 1$, and $f(x) = 0$ if $|x| \geq 2$.