Sample problems for the final exam (page 1 of 2)

Any problem may be altered!

Problem I For each of the following sets $E \subset \mathbb{R}$, determine the least upper bound $\sup E$ and the greatest lower bound $\inf E$. Briefly explain.

1. $E = \{(-1)^n(1+n^{-1}) : n \in \mathbb{N}\}$.
2. $E = \{(-1)^n(1-n^{-1}) : n \in \mathbb{N}\}$.
3. $E = \{n^2/2^n : n \in \mathbb{N}\}$.
4. $E = \{3^n/n! : n \in \mathbb{N}\}$.
5. $E = \{(2+(-1)^n)n/(n+3) : n \in \mathbb{N}\}$.
6. $E = \{m/(m+n) : m, n \in \mathbb{N}\}$.
7. $E = \{(m-n)/(m+n) : m, n \in \mathbb{N}\}$.
8. $E = \{m/(|m|+n) : m \in \mathbb{Z},\ n \in \mathbb{N}\}$.
9. $E = \{m/n + 4n/m : m, n \in \mathbb{N}\}$.
10. $E = \{mn/(4m^2+n^2) : m, n \in \mathbb{N}\}$.

Problem II Find limits of sequences, limits of functions, and test series for convergence. Briefly explain.

11. $\lim_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{2^n+3^n}$.
12. $\lim_{n\to\infty} \frac{n^3}{2^n}$.
13. $\lim_{n\to\infty} \sqrt[n]{3^n+5^n}$.
14. $\lim_{n\to\infty} (1+\sin 1000^{\text{o}})^n$.
15. $\lim_{n\to\infty} n \sin \frac{\pi}{n}$.
16. $\lim_{n\to\infty} \sin(2\pi\sqrt{n^2+1})$.
17. $\lim_{n\to\infty} \sin(\pi\sqrt{4n^2+n})$.
18. $\lim_{n\to\infty} \left(\cos\frac{1}{n}\right)^n$.
19. $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{2n}$.
20. $\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{3n}$.
21. $\lim_{x\to\infty} \frac{x^2+3x-4}{\sqrt{2x^4+1}}$.
22. $\lim_{x\to\infty} \left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right)$.
23. $\lim_{x\to\infty} \left(x+\sqrt[3]{1-x^3}\right)$.
24. $\lim_{x\to 0} \frac{\cos x - \cos 3x}{x^2}$.
25. $\lim_{x\to 0} \frac{\sin(x \sin 2x)}{x^2}$.
26. $\lim_{x\to 1} (1-x)\cot \pi x$.
27. $\lim_{x\to 0} \frac{1-e^{-x}}{\sin x}$.
28. $\lim_{x\to 0} (\cos x)^{1/x^2}$.
29. $\lim_{x\to 0} (2^x-1)^x$.
30. $\lim_{x\to\infty} x\log(1+x^{-1})$.
31. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$.
32. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2(n+1)}}$.
33. $\sum_{n=1}^{\infty} \frac{1}{n\,\log^3 n}$.
34. $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.
35. $\sum_{n=1}^{\infty} \frac{2^n\,n!}{n^n}$.
36. $\sum_{n=1}^{\infty} \frac{3^n\,n!}{n^n}$.
37. $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$.
38. $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$.
39. $\sum_{n=1}^{\infty} (-1)^n \frac{2n^2+1}{3n(n+1)}$.
40. $\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}}$.

Problem III Find the exact number of distinct real solutions of an equation

$$3x^5+5x^3-30x-22=0.$$

Prove your answer (the proof must not be computer-assisted).

Sample problems for the final exam (page 2 of 2)

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Problem IV Consider an infinitely differentiable function $f(x) = \arctan x$, $x \in \mathbb{R}$. Let $f^{(n)}$ denote the n-th derivative of f ($f^{(0)} = f$).

(i) Prove (by induction) that for any integer $n \geq 2$ there exist constants a_n, b_n such that $(1+x^2)f^{(n)}(x) + a_n x f^{(n-1)}(x) + b_n f^{(n-2)}(x) = 0$ for all $x \in \mathbb{R}$.

(ii) Find the constants a_n, b_n, $n = 2, 3, \dots$

(iii) Find all derivatives of the function f at 0.

Problem V Find indefinite integrals and evaluate definite integrals. Briefly explain.

41. $\displaystyle\int \frac{2x+3}{2x+1}\,dx.$

42. $\displaystyle\int \sqrt{3x-1}\,dx.$

43. $\displaystyle\int \frac{\sqrt{x}+\log x}{x}\,dx.$

44. $\displaystyle\int \sin 3x \cos 5x\,dx.$

45. $\displaystyle\int \frac{\sin x}{(1-\cos x)^3}\,dx.$

46. $\displaystyle\int \log^2 x\,dx.$

47. $\displaystyle\int \frac{\log x}{\sqrt{x}}\,dx.$

48. $\displaystyle\int \frac{e^x}{e^x+1}\,dx.$

49. $\displaystyle\int x^3 e^{-x^2}\,dx.$

50. $\displaystyle\int e^{\sqrt{x}}\,dx.$

51. $\displaystyle\int_1^4 \frac{1+\sqrt{x}}{x^2}\,dx.$

52. $\displaystyle\int_{-1}^1 \frac{x^5}{x+2}\,dx.$

53. $\displaystyle\int_0^4 \frac{1}{1+\sqrt{x}}\,dx.$

54. $\displaystyle\int_0^{\pi/2} x\cos x\,dx.$

55. $\displaystyle\int_0^{\pi} x^2 \sin x\,dx.$

56. $\displaystyle\int_{-\pi/4}^{\pi/4} \tan x\,dx.$

57. $\displaystyle\int_e^{e^2} \frac{1}{x\log x}\,dx.$

58. $\displaystyle\int_1^e \log x\,dx.$

59. $\displaystyle\int_0^1 x^2 e^{2x}\,dx.$

60. $\displaystyle\int_0^{\pi} e^x \sin x\,dx.$

61. $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx.$

62. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx.$

63. $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx.$

64. $\displaystyle\int_0^{\infty} \frac{\arctan x}{x^2+1}\,dx.$

65. $\displaystyle\int_0^{\pi/2} \cot x\,dx.$

66. $\displaystyle\int_0^{1/2} \frac{1}{x\log^2 x}\,dx.$

67. $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx.$

68. $\displaystyle\int_0^{\infty} xe^{-x}\,dx.$

69. $\displaystyle\int_0^{\infty} x^3 e^{-x}\,dx.$

70. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\sin x\,dx.$

Bonus Problem VI Which number is larger, 2019^{2020} or 2020^{2019}? Prove your answer (the proof must not be computer-assisted).

Bonus Problem VII Let $\{a_n\}$ be a sequence of distinct real numbers converging to a limit b. Suppose that a function f is infinitely differentiable at the point b and $f(a_n) = 0$ for all $n \in \mathbb{N}$. Prove that all derivatives of the function f at b are equal to 0.