MATH 409

Advanced Calculus I

Lecture 1: Axioms of a field.

Real line

Systematic study of the calculus of functions of one variable begins with the study of the domain of such functions, the set of real numbers $\mathbb{R}$ (real line).

The real line is a mathematical object rich with structure. This includes:

- algebraic structure (4 arithmetic operations);
- ordering (for any three points, one is located between the other two);
- metric structure (we can measure distances between points);
- continuity (we can get from one point to another in a continuous way, without jumps).

Axiomatic model

The study of the real line begins with formulation of an axiomatic model, which is to provide the solid foundation for all subsequent developments.

The axiomatic model of the real numbers shall be formulated using three **postulates**, each consisting of one or several axioms. To verify that the axiomatic model is adequate, one has to prove that the axioms are **consistent** (namely, there exists an object satisfying them) and **categorical** (namely, that object is, in a sense, unique).

The axioms are chosen among basic properties of the real numbers, which ensures consistency. Postulate 1 formalizes the algebraic structure, Postulate 2 formalizes the ordering, and Postulate 3 formalizes the continuous structure (the metric structure does not require a separate postulate; it can be formalized in terms of the other structures).

Field

The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ motivated the introduction of an abstract algebraic structure called a **field**. Informally, a field is a set with 4 arithmetic operations (addition, subtraction, multiplication, and division) that have roughly the same properties as those of real (or complex) numbers.

The notion of field is important for the linear algebra. Namely, a field is a set that can serve as a set of scalars for a vector space.

Formally, a field is a set F equipped with two binary operations, called **addition** and **multiplication** and denoted accordingly, that satisfy a number of axioms.

Field: formal definition

A **field** is a set F equipped with two operations, **addition** $F \times F \ni (a, b) \mapsto a + b \in F$ and **multiplication** $F \times F \ni (a, b) \mapsto a \cdot b \in F$, such that:

A1. $a + b = b + a$ for all $a, b \in F$.
A2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in F$.
A3. There exists an element of F, denoted 0, such that $a + 0 = 0 + a = a$ for all $a \in F$.
A4. For any $a \in F$ there exists an element of F, denoted $-a$, such that $a + (-a) = (-a) + a = 0$.

M1. $a \cdot b = b \cdot a$ for all $a, b \in F$.
M2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in F$.
M3. There exists an element of F different from 0, denoted 1, such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in F$.
M4. For any $a \in F$, $a \neq 0$ there exists an element of F, denoted a^{-1}, such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

AM. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in F$.

Alternative notation: $a \cdot b$ can be denoted ab (if it does not create confusion).

Subtraction and **division** in the field F are defined as follows: $a - b = a + (-b)$, $a/b = a \cdot b^{-1}$.

Postulate 1. The set of real numbers $\mathbb{R}$ is a field.

Other examples of fields:

- Complex numbers $\mathbb{C}$.
- Rational numbers $\mathbb{Q}$.
- $\mathbb{R}(x)$: rational functions $f(x)$ in variable x with real coefficients; $f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$, where $a_i, b_j \in \mathbb{R}$ and $b_m \neq 0$.
- $\mathbb{F}_2 = \{\bar{0}, \bar{1}\}$: field of two elements.

The operations are defined as follows: $\bar{0} + \bar{0} = \bar{1} + \bar{1} = \bar{0}$, $\bar{0} + \bar{1} = \bar{1} + \bar{0} = \bar{1}$, $\bar{0} \cdot \bar{0} = \bar{0} \cdot \bar{1} = \bar{1} \cdot \bar{0} = \bar{0}$, $\bar{1} \cdot \bar{1} = \bar{1}$.

Basic properties of fields

- The zero 0 is unique.

Suppose z_1 and z_2 are both zeros, that is, $a + z_1 = z_1 + a = a$ and $a + z_2 = z_2 + a = a$ for all $a \in F$. Then $z_1 + z_2 = z_2$ and $z_1 + z_2 = z_1$. Hence $z_1 = z_2$.

- For any $a \in F$, the negative $-a$ is unique.

Suppose b_1 and b_2 are both negatives of a. Let us compute the sum $b_1 + a + b_2$ in two ways:

$$(b_1 + a) + b_2 = 0 + b_2 = b_2,$$
$$b_1 + (a + b_2) = b_1 + 0 = b_1.$$

By associativity of the addition, $b_1 = b_2$.

- The unity 1 is unique.
- For any $a \neq 0$, the inverse a^{-1} is unique.