MATH 409

Advanced Calculus I

# Lecture 3:
# Supremum and infimum.
# Completeness axiom.

# Bounds

Suppose $X$ is a set with a strict linear order $\prec$.

*Definition.* Let $E \subset X$ be a nonempty set and $M \in X$. We say that $M$ is an **upper bound** of the set $E$ if $x \preceq M$ for all $x \in E$. Similarly, $M$ is a **lower bound** of the set $E$ if $x \succeq M$ for all $x \in E$.

We say that the set $E$ is **bounded above** if it admits an upper bound and **bounded below** if it admits a lower bound. The set $E$ is called **bounded** if it is bounded above and below.

An element $M \in X$ is called the **maximum** of a set $E \subset X$ and denoted $\max E$ if (i) $M$ is an upper bound of $E$ and (ii) $M$ belongs to $E$.

Similarly, $M$ is called the **minimum** of the set $E$ and denoted $\min E$ if (i) $M$ is a lower bound of $E$ and (ii) $M$ belongs to $E$.

# Supremum and infimum

*Definition.* An element $M \in X$ is called the **supremum** (or the **least upper bound**) of the set $E$ and denoted $\sup E$ if (i) $M$ is an upper bound of $E$ and (ii) $M \preceq M_+$ for any upper bound $M_+$ of $E$.

Similarly, $M$ is called the **infimum** (or the **greatest lower bound**) of the set $E$ and denoted $\inf E$ if (i) $M$ is a lower bound of $E$ and (ii) $M \succeq M_-$ for any lower bound $M_-$ of $E$.

If $\max E$ exists then it is also $\sup E$. However $\sup E$ may exist even if $\max E$ does not. Similarly, $\inf E = \min E$ whenever $\min E$ exists.

*Examples.* • $X = \mathbb{R}$, $E = [0, 1]$.

$\max E = \sup E = 1$, $\min E = \inf E = 0$.

• $X = \mathbb{R}$, $E = (0, 1)$.

$\sup E = 1$, $\inf E = 0$, $\max E$ and $\min E$ do not exist.

# Axioms of real numbers

*Definition.* The set $\mathbb{R}$ of real numbers is a set satisfying the following postulates:

**Postulate 1.** $\mathbb{R}$ is a field.

**Postulate 2.** There is a strict linear order $<$ on $\mathbb{R}$ that makes it into an ordered field.

**Postulate 3 (Completeness Axiom).**
If a nonempty subset $E \subset \mathbb{R}$ is bounded above, then $E$ has a supremum.

**Theorem 1** Suppose $X$ and $Y$ are nonempty subsets of $\mathbb{R}$ such that $a \leq b$ for all $a \in X$ and $b \in Y$. Then there exists $c \in \mathbb{R}$ such that $a \leq c$ for all $a \in X$ and $c \leq b$ for all $b \in Y$.

*Proof:* The set $X$ is bounded above as any element of $Y$ is an upper bound of $X$. By Completeness Axiom, $\sup X$ exists. We have $a \leq \sup X$ for all $a \in X$ since $\sup X$ is an upper bound of $X$. Besides, $\sup X \leq b$ for any $b \in Y$ since $b$ is an upper bound of $X$ while $\sup X$ is the least upper bound.

**Theorem 2** If a nonempty subset $E \subset \mathbb{R}$ is bounded below, then $E$ has an infimum.

*Proof:* Let $X$ denote the set of all lower bounds of $E$. Then $a \leq b$ for all $a \in X$ and $b \in E$. Since $E$ is bounded below, the set $X$ is not empty. By Theorem 1, there exists $c \in \mathbb{R}$ such that $a \leq c$ for all $a \in X$ and $c \leq b$ for all $b \in E$. That is, $c$ is a lower bound of $E$ and an upper bound of $X$. It follows that $c = \inf E$.

# Dedekind cuts

Suppose $X$ is a set with a strict linear order $\prec$.

*Definition.* A **Dedekind cut** of $X$ is a partition $X = L \cup R$, where $L$ and $R$ are disjoint sets such that $x \prec y$ for all $x \in L$ and $y \in R$.

For any Dedekind cut, there are 4 possibilities.

**Case 1.** $\max L$ exists while $\min R$ does not.
In this case, $L = \{x \in X \mid x \preceq x_0\}$ and $R = \{x \in X \mid x \succ x_0\}$, where $x_0 = \max L$.

*Example.* $X = \mathbb{R}$, $L = (-\infty, 0]$, $R = (0, \infty)$.

**Case 2.** $\min R$ exists while $\max L$ does not.
In this case, $L = \{x \in X \mid x \prec y_0\}$ and $R = \{x \in X \mid x \succeq y_0\}$, where $y_0 = \min R$.

*Example.* $X = \mathbb{R}$, $L = (-\infty, 0)$, $R = [0, \infty)$.

**Case 3.** Both $\max L$ and $\min R$ exist.
In this case, $L = \{x \in X \mid x \preceq x_0\}$ and
$R = \{x \in X \mid x \succeq y_0\}$, where $x_0 = \max L$, $y_0 = \min R$.
*Example.* $X = (-\infty, 0] \cup [1, \infty)$, $L = (-\infty, 0]$, $R = [1, \infty)$.

**Case 4.** Neither $\max L$ nor $\min R$ exists.

*Examples.* • $X = \mathbb{R} \setminus \{0\}$, $L = (-\infty, 0)$, $R = (0, \infty)$.
• $X = \mathbb{Q}$, $L = \{x \in \mathbb{Q} \mid x \leq 0 \text{ or } x^2 < 2\}$,
$R = \{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 > 2\}$.

Cases 1 and 2 are normal cuts. Case 3 shows a "gap" in the order (from $x_0$ to $y_0$). Case 4 shows a "hole" in the order (between $L$ and $R$).

Case 3 is not possible for ordered fields as $x_0 \prec \frac{1}{2}(x_0 + y_0) \prec y_0$.
Case 4 is ruled out by the Completeness Axiom.

*Remark.* Richard Dedekind used his cuts to construct real numbers from the rationals.

# Natural, integer, and rational numbers

Postulate 1 guarantees that $\mathbb{R}$ contains numbers 0 and 1. Then we can define natural numbers $2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$, and so on... It was proved in the previous lecture that $0 < 1$. Repeatedly adding 1 to both sides of this inequality, we obtain $0 < 1 < 2 < 3 < \dots$ In particular, all these numbers are distinct.

However the entire set of natural numbers can only be defined in an implicit way.

*Definition.* A set $E \subset \mathbb{R}$ is called **inductive** if $1 \in E$ and, for any real number $x$, $x \in E$ implies $x + 1 \in E$. The set $\mathbb{N}$ of **natural numbers** is the smallest inductive subset of $\mathbb{R}$ (namely, it is the intersection of all inductive subsets of $\mathbb{R}$).

The set of **integers** is defined as $\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$.
The set of **rationals** is defined as $\mathbb{Q} = \{m/n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$.

# Archimedean Principle

**Theorem (Archimedean Principle)** For any real number $\varepsilon > 0$ there exists a natural number $n$ such that $n\varepsilon > 1$.

*Remark.* Archimedean Principle means that $\mathbb{R}$ contains no **infinitesimal** (i.e., infinitely small) numbers other than 0.

*Proof:* In the case $\varepsilon > 1$, we can take $n = 1$. Now assume $\varepsilon \leq 1$. Let $E$ be the set of all natural numbers $n$ such that $n\varepsilon \leq 1$. Observe that $E$ is nonempty ($1 \in E$) and bounded above ($1/\varepsilon$ is an upper bound). By Completeness Axiom, $m = \sup E$ exists. By definition of $\sup E$, there exists $n \in E$ such that $n > m - 1/2$ (as otherwise $m - 1/2$ would be an upper bound for $E$). Then $n + 1$ is a natural number and $n + 1 > m + 1/2 > m$. It follows that $n + 1$ is not in $E$. Consequently, $(n + 1)\varepsilon > 1$. ∎

**Corollary** For any $a, b > 0$ there exists a natural number $n$ such that $na > b$.