MATH 409

Advanced Calculus I

Lecture 4:
Archimedean principle.
Mathematical induction.
Binomial formula.

Natural, integer, and rational numbers

Definition. A set $E \subset \mathbb{R}$ is called **inductive** if $1 \in E$ and, for any real number x, $x \in E$ implies $x + 1 \in E$. The set $\mathbb{N}$ of **natural numbers** is the smallest inductive subset of $\mathbb{R}$.

Remark. The set $\mathbb{N}$ is well defined. Namely, it is the intersection of all inductive subsets of $\mathbb{R}$.

The set of **integers** is defined as

$$\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}.$$

The set of **rationals** is defined as

$$\mathbb{Q} = \{m/n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}.$$

Archimedean Principle

Theorem (Archimedean Principle) For any real number $\varepsilon > 0$ there exists a natural number n such that $n\varepsilon > 1$.

Remark. Archimedean Principle means that $\mathbb{R}$ contains no **infinitesimal** (i.e., infinitely small) numbers other than 0.

Proof: In the case $\varepsilon > 1$, we can take $n = 1$. Now assume $\varepsilon \leq 1$. Let E be the set of all natural numbers n such that $n\varepsilon \leq 1$. Observe that E is nonempty ($1 \in E$) and bounded above ($1/\varepsilon$ is an upper bound). By Completeness Axiom, $m = \sup E$ exists. By definition of $\sup E$, there exists $n \in E$ such that $n > m - 1/2$ (as otherwise $m - 1/2$ would be an upper bound for E). Then $n + 1$ is a natural number and $n + 1 > m + 1/2 > m$. It follows that $n + 1$ is not in E. Consequently, $(n + 1)\varepsilon > 1$. ■

Corollary For any $a, b > 0$ there exists a natural number n such that $na > b$.

Basic properties of the natural numbers

- 1 is the least natural number.

The interval $[1, \infty)$ is an inductive set. Hence $\mathbb{N} \subset [1, \infty)$.

- If $n \in \mathbb{N}$, then $n - 1 \in \mathbb{N} \cup \{0\}$.

Let E be the set of all $n \in \mathbb{N}$ such that $n - 1 \in \mathbb{N} \cup \{0\}$. Then $1 \in E$ as $1 - 1 = 0$. Besides, for any $n \in E$ we have $(n + 1) - 1 = n \in \mathbb{N}$ so that $n + 1 \in E$. Therefore E is an inductive set. Then $\mathbb{N} \subset E$, which implies that $E = \mathbb{N}$.

- If $n \in \mathbb{N}$, then the open interval $(n - 1, n)$ contains no natural numbers.

Let E be the set of all $n \in \mathbb{N}$ such that $(n - 1, n) \cap \mathbb{N} = \emptyset$. Then $1 \in E$ as $\mathbb{N} \subset [1, \infty)$. Now assume $n \in E$ and take any $x \in (n, n + 1)$. We have $x - 1 \neq 0$ since $x > n \geq 1$, and $x - 1 \notin \mathbb{N}$ since $x - 1 \in (n - 1, n)$. By the above, $x \notin \mathbb{N}$. Thus E is an inductive set, which implies that $E = \mathbb{N}$.

Principle of well-ordering

Definition. Suppose X is a set endowed with a strict linear order $\prec$. We say that a subset $Y \subset X$ is **well-ordered** with respect to $\prec$ if any nonempty subset of Y has a least element.

Theorem The set $\mathbb{N}$ is well-ordered with respect to the natural ordering of the real line $\mathbb{R}$.

Proof: Let E be an arbitrary nonempty subset of $\mathbb{N}$. The set E is bounded below since 1 is a lower bound of $\mathbb{N}$. Therefore $m = \inf E$ exists. Since m is a lower bound of E while $m+1$ is not, we can find $n \in E$ such that $m \leq n < m+1$. As shown before, the interval $(n-1, n)$ is disjoint from $\mathbb{N}$. Then $(-\infty, n) = (-\infty, m) \cup (n-1, n)$ is disjoint from E, which implies that n is a lower bound of E. Hence $n \leq \inf E = m$ so that $n = m = \inf E$. Thus n is the least element of E.

Principle of mathematical induction

Theorem Let $P(n)$ be an assertion depending on a natural variable n. Suppose that

- $P(1)$ holds,
- whenever $P(k)$ holds, so does $P(k+1)$.

Then $P(n)$ holds for all $n \in \mathbb{N}$.

Proof: Let E be the set of all natural numbers n such that $P(n)$ holds. Clearly, E is an inductive set. Therefore $\mathbb{N} \subset E$, which implies that $E = \mathbb{N}$. ■

Remark. The assertion $P(1)$ is called the **basis of induction**. The implication $P(k) \implies P(k+1)$ is called the **induction step**.

Examples of assertions $P(n)$:

(a) $1 + 2 + \cdots + n = n(n+1)/2$,
(b) $n(n+1)(n+2)$ is divisible by 6,
(c) $n = 2p + 3q$ for some $p, q \in \mathbb{Z}$.

Strong induction

Theorem Let $P(n)$ be an assertion depending on a natural variable n. Suppose that $P(n)$ holds whenever $P(k)$ holds for all natural $k < n$. Then $P(n)$ holds for all $n \in \mathbb{N}$.

Remark. For $n = 1$, the assumption of the theorem means that $P(1)$ holds unconditionally. For $n = 2$, it means that $P(1)$ implies $P(2)$. For $n = 3$, it means that $P(1)$ and $P(2)$ imply $P(3)$. And so on...

Proof of the theorem: For any natural number n we define new assertion $Q(n) =$"$P(k)$ holds for any natural $k \leq n$". Then $Q(1)$ is equivalent to $P(1)$, in particular, it holds. By assumption, $Q(n)$ implies $P(n+1)$ for any $n \in \mathbb{N}$. Moreover, $Q(n+1)$ holds if and only if both $Q(n)$ and $P(n+1)$ hold. Therefore, $Q(n)$ implies $Q(n+1)$ for all $n \in \mathbb{N}$. By the principle of mathematical induction, $Q(n)$ holds for all $n \in \mathbb{N}$. Then $P(n)$ holds for all $n \in \mathbb{N}$ as well.

Well-ordering and induction

Principle of well-ordering:
The set $\mathbb{N}$ is well-ordered, that is, any nonempty subset of $\mathbb{N}$ has a least element.

Principle of mathematical induction:
Let $P(n)$ be an assertion depending on a natural variable n. Suppose that $P(1)$ holds and $P(k)$ implies $P(k+1)$ for any $k \in \mathbb{N}$. Then $P(n)$ holds for all $n \in \mathbb{N}$.

Induction with a different base:
Let $P(n)$ be an assertion depending on an integer variable n. Suppose that $P(n_0)$ holds for some $n_0 \in \mathbb{Z}$ and $P(k)$ implies $P(k+1)$ for any $k \geq n_0$. Then $P(n)$ holds for all $n \geq n_0$.

Strong induction: Let $P(n)$ be an assertion depending on a natural variable n. Suppose that $P(n)$ holds whenever $P(k)$ holds for all natural $k < n$. Then $P(n)$ holds for all $n \in \mathbb{N}$.

Inductive definition

The principle of mathematical induction allows to define mathematical objects inductively (that is, recursively).

Examples of inductive definitions:

- Power a^n of a number

Given a real number a, we let $a^0 = 1$ and $a^n = a^{n-1}a$ for any $n \in \mathbb{N}$.

- Factorial $n!$

We let $0! = 1$ and $n! = (n-1)! \cdot n$ for any $n \in \mathbb{N}$.

- Fibonacci numbers $F_1, F_2, \ldots$

We let $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for any $n \geq 3$.

Binomial coefficients

Definition. For any integers n and k, $0 \leq k \leq n$, we define the **binomial coefficient** $\binom{n}{k}$ (n choose k) by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

If $k > 0$ then $\displaystyle\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k}$.

"n choose k" refers to the fact that $\binom{n}{k}$ is the number of all k-element subsets of an n-element set.

Lemma $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ for all n and k, $1 \leq k \leq n$.

Proof: $\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}$

$= \frac{n!}{(k-1)!\,(n-k)!}\left(\frac{1}{n-k+1} + \frac{1}{k}\right) = \frac{n!}{(k-1)!\,(n-k)!} \cdot \frac{n+1}{k(n-k+1)}$

$= \frac{(n+1)!}{k!\,(n-k+1)!} = \binom{n+1}{k}$.

Pascal's triangle

The formula $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ allows to compute the binomial coefficients recursively. Usually the results are formatted as a triangular array called **Pascal's triangle**. Namely, $\binom{n}{k}$ is the k-th number in the n-th row of the triangle (the numbering of rows and elements in a row starts from 0).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Binomial formula

Theorem For any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

In particular, $(a+b)^2 = a^2 + 2ab + b^2$,
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

The coefficients in the binomial formula are consecutive numbers in the n-th row of Pascal's triangle.

Theorem For any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Proof: The proof is by induction on n. In the case $n = 1$, the formula is trivial: $(a+b)^1 = \binom{1}{0} a + \binom{1}{1} b$. Now assume that the formula holds for a particular value of n. Then

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n-k+1} b^k$$

$$= \binom{n}{0} a^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) a^{n-k+1} b^k + \binom{n}{n} b^{n+1}$$

$$= \binom{n+1}{0} a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + \binom{n+1}{n+1} b^{n+1},$$

which completes the induction step.