MATH 409

Advanced Calculus I

Lecture 7a:
Absolute value.
Metric spaces.

Absolute value

Definition. The **absolute value** (or **modulus**) of a real number a, denoted $|a|$, is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Properties of the absolute value:

- $|a| \geq 0$.
- $|a| = 0$ if and only if $a = 0$.
- $|-a| = |a|$.
- $|a| = \max(a, -a)$.
- $-|a| \leq a \leq |a|$.

- $|a| = M \iff M \geq 0$ and $a = \pm M$.
- If $M > 0$, then $|a| < M \iff -M < a < M$.
- $|ab| = |a| \cdot |b|$.

We have that $|a| = a$ or $-a$. Likewise, $|b| = b$ or $-b$. Therefore $|a| \cdot |b|$ is one of the numbers ab, $(-a)b$, $a(-b)$ or $(-a)(-b)$. We know that $(-a)(-b) = ab$ and $(-a)b = a(-b) = -ab$. Hence $|a| \cdot |b| = ab$ or $-ab$. Besides, $|a| \cdot |b| \geq 0$. Thus $|a| \cdot |b| = |ab|$.

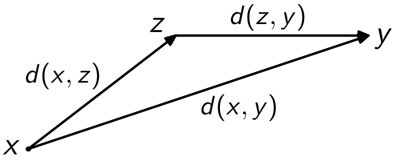
- $|a + b| \leq |a| + |b|$.

Since $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, it follows that $-(|a| + |b|) \leq a + b \leq |a| + |b|$, which is equivalent to $|a + b| \leq |a| + |b|$.

Metric space

Definition. Given a nonempty set X, a **metric** (or **distance function**) on X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following conditions:

- **(positivity)** $d(x, y) \geq 0$ for all $x, y \in X$; moreover, $d(x, y) = 0$ if and only if $x = y$;
- **(symmetry)** $d(x, y) = d(y, x)$ for all $x, y \in X$;
- **(triangle inequality)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.



A set endowed with a metric is called a **metric space**.

Theorem The function $d(x,y) = |y - x|$ is a metric on the real line $\mathbb{R}$.

Proof: We have $|y - x| \geq 0$ for all $x, y \in \mathbb{R}$. Moreover, $|y - x| = 0$ only if $y - x = 0$, which is equivalent to $x = y$. This proves positivity.

Symmetry follows since $x - y = -(y - x)$ and $|-a| = |a|$ for all $a \in \mathbb{R}$.

Finally, $d(x,y) = |y - x| = |(y - z) + (z - x)|$
$\leq |y - z| + |z - x| = d(z,y) + d(x,z)$.

Other examples of metric spaces

- *Euclidean space*

$X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$.

- *Normed vector space*

X: vector space with a norm $\|\cdot\|$, $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$.

- *Discrete metric space*

X: any nonempty set, $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

- *Space of sequences*

X: set of all infinite words $x = x_1 x_2 \dots$ over a finite alphabet;
$d(x, y) = 2^{-n}$ if $x_i = y_i$ for $1 \leq i \leq n$ while $x_{n+1} \neq y_{n+1}$,
$d(x, y) = 0$ if $x_i = y_i$ for all $i \geq 1$.