MATH 409

Advanced Calculus I

Lecture 7b:
Limit of a sequence.

Convergence of a sequence

A **sequence** of elements of a set X is a function $f : \mathbb{N} \to X$.
Notation: $x_1, x_2, \dots,$ where $x_n = f(n)$, or $\{x_n\}_{n\in\mathbb{N}}$, or $\{x_n\}$.

Suppose X is a metric space with a distance function d.

Definition. Sequence $\{x_n\}$ of elements of X is said to **converge** to an element a if for any real number $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d(x_n, a) < \varepsilon$ whenever $n \geq N$. The point a is called the **limit** of the sequence $\{x_n\}$.
Notation: $\lim_{n\to\infty} x_n = a$, or $x_n \to a$ as $n \to \infty$.

A sequence is called **convergent** if it has a limit and **divergent** otherwise.

Given $a \in X$ and $\varepsilon > 0$, let $B_\varepsilon(a) = \{x \in X \mid d(x, a) < \varepsilon\}$ be the set of all points in X at distance less than ε from a. This set is called the ε-**neighborhood** of the point a or the **ball** of radius ε centered at a.

Convergence of a sequence in $\mathbb{R}$

Definition. Sequence $\{x_n\}$ of real numbers is said to **converge** to a real number a if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N$. The number a is called the **limit** of $\{x_n\}$. Notation: $\lim_{n\to\infty} x_n = a$, or $x_n \to a$ as $n \to \infty$.

A sequence is called **convergent** if it has a limit and **divergent** otherwise.

The condition $|x_n - a| < \varepsilon$ is equivalent to $a - \varepsilon < x_n < a + \varepsilon$ or to $x_n \in (a - \varepsilon, a + \varepsilon)$. Hence the ε-**neighborhood** of the point a is the interval $(a - \varepsilon, a + \varepsilon)$.

Proposition $x_n \to a$ as $n \to \infty$ if and only if any ε-neighborhood of the point a contains all but finitely many terms of the sequence $\{x_n\}$.

Examples

- The sequence $\{1/n\}_{n\in\mathbb{N}}$ converges to 0.

By the Archimedean Principle, for any $\varepsilon > 0$ there exists a natural number N such that $N\varepsilon > 1$. Then for any integer $n \geq N$ we have $n\varepsilon \geq N\varepsilon > 1$ so that $1/n < \varepsilon$. Since $1/n > 0$, we obtain $|1/n| < \varepsilon$ for all $n \geq N$.

- Constant sequence $\{x_n\}$, where $x_n = a$ for some $a \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Since $|x_n - a| = 0$ for all $n \in \mathbb{N}$, the sequence converges to a.

- Sequence $\{(-1)^n\}_{n\in\mathbb{N}}$, i.e., $-1, 1, -1, 1, \ldots,$ is divergent.

- Sequence $\{n\}_{n\in\mathbb{N}}$, i.e., $1, 2, 3, 4, \ldots,$ is divergent.