MATH 409

Advanced Calculus I

Lecture 8:
Properties of limits.
Divergent sequences.

Convergence of a sequence

A **sequence** of elements of a set X is a function $f : \mathbb{N} \to X$.
Notation: $x_1, x_2, \dots,$ where $x_n = f(n)$, or $\{x_n\}_{n\in\mathbb{N}}$, or $\{x_n\}$.

Definition. Sequence $\{x_n\}$ of real numbers is said to **converge** to a real number a if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N$. The number a is called the **limit** of $\{x_n\}$. Notation: $\lim_{n\to\infty} x_n = a$, or $x_n \to a$ as $n \to \infty$.

A sequence is called **convergent** if it has a limit and **divergent** otherwise.

The condition $|x_n - a| < \varepsilon$ is equivalent to $a - \varepsilon < x_n < a + \varepsilon$ or to $x_n \in (a - \varepsilon, a + \varepsilon)$. The interval $(a - \varepsilon, a + \varepsilon)$ is called the ε-**neighborhood** of the point a. The convergence $x_n \to a$ means that any ε-neighborhood of a contains all but finitely many elements of the sequence $\{x_n\}$.

Basic properties of convergent sequences

- The limit is unique.

Suppose a and b are distinct real numbers. Let $\varepsilon = \frac{1}{2}|b - a|$ ($=$ half distance from a to b). Then ε-neighborhoods of a and b are disjoint. Hence they cannot both contain all but finitely many elements of the same sequence.

- Any convergent sequence $\{x_n\}$ is **bounded**, which means that the set of its elements is bounded.

This follows from three facts: any ε-neighborhood is bounded, any finite set is bounded, and the union of two bounded sets is also bounded.

- Any **subsequence** converges to the same limit.

Here a subsequence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is any sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where $\{n_k\}$ is an increasing sequence of natural numbers (note that $n_k \geq k$).

Divergence to infinity

Definition. A sequence $\{x_n\}$ is said to **diverge to infinity** if for any $C \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n > C$ for all $n \geq N$.

Notation: $\lim_{n\to\infty} x_n = \infty$, or $x_n \to \infty$ as $n \to \infty$.

Observe that such a sequence is indeed divergent as it is not bounded.

Definition. A sequence $\{x_n\}$ is said to diverge to $-\infty$ if for any $C \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n < C$ for all $n \geq N$.

Notation: $\lim_{n\to\infty} x_n = -\infty$, or $x_n \to -\infty$ as $n \to \infty$.

Examples of divergent sequences

- The sequence $\{n\}_{n\in\mathbb{N}}$ diverges to ∞ (or $+\infty$).
- The sequence $\{-n^2\}_{n\in\mathbb{N}}$ diverges to $-\infty$.
- The sequence $\{x_n\}_{n\in\mathbb{N}}$, where $x_n = (-1)^n n$, does not diverge to ∞ or $-\infty$. It does leave any bounded set eventually. One writes $x_n \to \pm\infty$ as $n \to \infty$.
- The sequence $1, -1, 1, -1, 1, -1, \dots$ is divergent while being bounded.
- The sequence $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \dots$ is divergent while it is neither bounded nor diverging to $+\infty$ or $-\infty$.

Comparison Theorem

Theorem Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If $x_n \leq y_n$ for all sufficiently large n, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Proof: Let $a = \lim_{n\to\infty} x_n$ and $b = \lim_{n\to\infty} y_n$. Assume to the contrary that $a > b$. Then $\varepsilon = (a-b)/2$ is a positive number. Hence there exists a natural number N such that $|x_n - a| < \varepsilon$ and $|y_n - b| < \varepsilon$ for all $n \geq N$. In particular, $y_n < b + \varepsilon$ and $a - \varepsilon < x_n$ for $n \geq N$. However $b + \varepsilon = a - \varepsilon = (a+b)/2$, which implies that $y_n < x_n$ for all $n \geq N$, a contradiction.

Corollary If all elements of a convergent sequence $\{x_n\}$ belong to a closed interval $[a, b]$, then the limit belongs to $[a, b]$ as well.

Squeeze Theorem

Theorem Suppose $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ are three sequences of real numbers such that

$$x_n \leq w_n \leq y_n \quad \text{for all sufficiently large } n.$$

If the sequences $\{x_n\}$ and $\{y_n\}$ both converge to the same limit a, then $\{w_n\}$ converges to a as well.

Proof: Since $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = a$, for any $\varepsilon > 0$ there exist natural numbers N_1 and N_2 such that $a - \varepsilon < x_n < a + \varepsilon$ for all $n \geq N_1$ and $a - \varepsilon < y_n < a + \varepsilon$ for all $n \geq N_2$. Besides, there exists $N_0 \in \mathbb{N}$ such that $x_n \leq w_n \leq y_n$ for all $n \geq N_0$. Set $N = \max(N_0, N_1, N_2)$. Then for any natural number $n \geq N$ we have $a - \varepsilon < x_n \leq w_n \leq y_n < a + \varepsilon$, which implies that $a - \varepsilon < w_n < a + \varepsilon$. Thus $\lim\limits_{n\to\infty} w_n = a$.