MATH 409

Advanced Calculus I

Lecture 9:
Algebra of limits.

Convergence and arithmetic operations

Theorem Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences of real numbers. Then the sequences $\{x_n + y_n\}_{n\in\mathbb{N}}$ and $\{x_n - y_n\}_{n\in\mathbb{N}}$ are also convergent.

Moreover, if $a = \lim_{n\to\infty} x_n$ and $b = \lim_{n\to\infty} y_n$, then

$$\lim_{n\to\infty}(x_n + y_n) = a + b \text{ and } \lim_{n\to\infty}(x_n - y_n) = a - b.$$

Proof: Since $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$, for any $\varepsilon > 0$ there exists a natural number N such that $|x_n - a| < \varepsilon/2$ and $|y_n - b| < \varepsilon/2$ for all $n \geq N$. Then for any $n \geq N$ we obtain

$$\begin{aligned}
|(x_n + y_n) - (a + b)| &= |(x_n - a) + (y_n - b)| \\
&\leq |x_n - a| + |y_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \\
|(x_n - y_n) - (a - b)| &= |(x_n - a) + (b - y_n)| \\
&\leq |x_n - a| + |b - y_n| = |x_n - a| + |y_n - b| < \varepsilon.
\end{aligned}$$

Thus $x_n + y_n \to a + b$ and $x_n - y_n \to a - b$ as $n \to \infty$.

Theorem Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences of real numbers. Then the sequence $\{x_n y_n\}_{n\in\mathbb{N}}$ is also convergent. Moreover, if $a = \lim_{n\to\infty} x_n$ and $b = \lim_{n\to\infty} y_n$, then $\lim_{n\to\infty} x_n y_n = ab$.

Proof: Since $x_n \to a$ and $y_n \to b$ as $n \to \infty$, for any $\delta > 0$ there exists $N(\delta) \in \mathbb{N}$ such that $|x_n - a| < \delta$ and $|y_n - b| < \delta$ for all $n \geq N(\delta)$. Then for any $n \geq N(\delta)$ we obtain

$$\begin{aligned}
|x_n y_n - ab| &= |x_n y_n - a y_n + a y_n - ab| = |(x_n - a) y_n + a(y_n - b)| \\
&= |(x_n - a) y_n - (x_n - a) b + (x_n - a) b + a(y_n - b)| \\
&= |(x_n - a)(y_n - b) + (x_n - a) b + a(y_n - b)| \\
&\leq |(x_n - a)(y_n - b)| + |(x_n - a) b| + |a(y_n - b)| \\
&= |x_n - a|\,|y_n - b| + |b|\,|x_n - a| + |a|\,|y_n - b| \\
&< \delta^2 + (|a| + |b|)\delta.
\end{aligned}$$

Now, given $\varepsilon > 0$, we set $\delta = \min(1, (1 + |a| + |b|)^{-1}\varepsilon)$. Then $\delta > 0$ and $\delta^2 + (|a| + |b|)\delta \leq (1 + |a| + |b|)\delta \leq \varepsilon$. By the above, $|x_n y_n - ab| < \varepsilon$ for all $n \geq N(\delta)$.

Theorem Suppose that a sequence $\{x_n\}$ converges to some $a \in \mathbb{R}$. If $a \neq 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence $\{x_n^{-1}\}_{n\in\mathbb{N}}$ converges to a^{-1}.

Proof: Since $x_n \to a$ as $n \to \infty$, for any $\delta > 0$ there exists $N(\delta) \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n \geq N(\delta)$.

Given $\varepsilon > 0$, we set $\delta = \min(|a|/2, |a|^2\varepsilon/2)$. Then for any $n \geq N(\delta)$ we have $|x_n - a| < |a|/2$. Since

$$|a| \leq |a - x_n| + |x_n| = |x_n - a| + |x_n|,$$

it follows that $|x_n| \geq |a| - |x_n - a| > |a| - |a|/2 = |a|/2$.

Furthermore, for any $n \geq N(\delta)$ we obtain

$$\left|\frac{1}{x_n} - \frac{1}{a}\right| = \left|\frac{a - x_n}{ax_n}\right| = \frac{|x_n - a|}{|a|\,|x_n|} \leq \frac{2|x_n - a|}{|a|^2} < \frac{2\delta}{|a|^2} \leq \varepsilon.$$

We conclude that $1/x_n \to 1/a$ as $n \to \infty$.

Corollary 1 If $\lim_{n\to\infty} x_n = a$, then $\lim_{n\to\infty} cx_n = ca$ for any $c \in \mathbb{R}$.

Corollary 2 If $\lim_{n\to\infty} x_n = a$, then $\lim_{n\to\infty} (-x_n) = -a$.

Corollary 3 If $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$, and, moreover, $b \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n/y_n = a/b$.

Proof: Since $b \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, it follows that $y_n^{-1} \to b^{-1}$ as $n \to \infty$. Since $x_n/y_n = x_n y_n^{-1}$ for all $n \in \mathbb{N}$, we obtain that $x_n/y_n \to ab^{-1} = a/b$ as $n \to \infty$.

Example

- $\lim_{n\to\infty} \dfrac{(1+2n)^2}{1+2n^2} = ?$

$$\frac{(1+2n)^2}{1+2n^2} = \frac{(1+2n)^2/n^2}{(1+2n^2)/n^2} = \frac{(1/n+2)^2}{(1/n)^2+2} \quad \text{for all } \; n \in \mathbb{N}.$$

Since $1/n \to 0$ as $n \to \infty$, it follows that

$$1/n + 2 \to 0 + 2 = 2 \quad \text{as} \quad n \to \infty,$$

$$(1/n+2)^2 \to 2^2 = 4 \quad \text{as} \quad n \to \infty,$$

$$(1/n)^2 \to 0^2 = 0 \quad \text{as} \quad n \to \infty,$$

$$(1/n)^2 + 2 \to 0 + 2 = 2 \quad \text{as} \quad n \to \infty,$$

and, finally, $\dfrac{(1/n+2)^2}{(1/n)^2+2} \to \dfrac{4}{2} = 2$ as $n \to \infty$.

More properties of limits

Theorem If a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to some $a \in \mathbb{R}$, then the sequence $\{|x_n|\}_{n\in\mathbb{N}}$ converges to $|a|$.

Proof: We have $||x| - |y|| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Hence $||x_n| - |a|| < \varepsilon$ whenever $|x_n - a| < \varepsilon$. It follows that $|x_n| \to |a|$ as $n \to \infty$ whenever $x_n \to a$ as $n \to \infty$.

Theorem If $x_n \to a$ and $y_n \to b$ as $n \to \infty$, then $\max(x_n, y_n) \to \max(a, b)$ and $\min(x_n, y_n) \to \min(a, b)$ as $n \to \infty$.

Idea of the proof: $\max(x_n, y_n) = \frac{1}{2}(x_n + y_n) + \frac{1}{2}|x_n - y_n|$,
$\min(x_n, y_n) = \frac{1}{2}(x_n + y_n) - \frac{1}{2}|x_n - y_n|$.