MATH 409

Advanced Calculus I

Lecture 10:
Monotonic sequences.

Monotonic sequences

Definition. A sequence $\{x_n\}$ is called **nondecreasing** if $x_1 \leq x_2 \leq x_3 \leq \ldots$ or, to be precise, $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. It is called (strictly) **increasing** if $x_1 < x_2 < x_3 < \ldots$, that is, $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.

Likewise, the sequence $\{x_n\}$ is called **nonincreasing** if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. It is called (strictly) **decreasing** if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.

All of the above sequences are called **monotonic**.

Examples:

- the sequence $\{1/n\}$ is decreasing;
- the sequence $1, 1, 2, 2, 3, 3, \ldots$ is nondecreasing, but not strictly increasing;
- the sequence $-1, 1, -1, 1, -1, 1, \ldots$ is neither nondecreasing nor nonincreasing;
- a constant sequence is both nondecreasing and nonincreasing.

Theorem Any nondecreasing sequence converges to a limit if bounded, and diverges to $+\infty$ otherwise.

Proof: Let $\{x_n\}$ be a nondecreasing sequence. First consider the case when $\{x_n\}$ is bounded. In this case, the set E of all elements occurring in the sequence is bounded. Then $\sup E$ exists. We claim that $x_n \to \sup E$ as $n \to \infty$. Take any $\varepsilon > 0$. Then $\sup E - \varepsilon$ is not an upper bound of E. Hence there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} > \sup E - \varepsilon$. Since the sequence is nondecreasing, we have $x_n \geq x_{n_0} > \sup E - \varepsilon$ for all $n \geq n_0$. At the same time, $x_n \leq \sup E$ for all $n \in \mathbb{N}$. Therefore $|x_n - \sup E| < \varepsilon$ for all $n \geq n_0$, which proves the claim.

Now consider the case when the sequence $\{x_n\}$ is not bounded. Note that the set E is bounded below (as x_1 is a lower bound). Hence E is not bounded above. Then for any $C \in \mathbb{R}$ there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} > C$. It follows that $x_n \geq x_{n_0} > C$ for all $n \geq n_0$. Thus $\{x_n\}$ diverges to $+\infty$.

Theorem Any nonincreasing sequence converges to a limit if bounded, and diverges to $-\infty$ otherwise.

Proof: Let $\{x_n\}$ be a nonincreasing sequence. Then the sequence $\{-x_n\}$ is nondecreasing since the inequality $a \geq b$ is equivalent to $-a \leq -b$ for all $a, b \in \mathbb{R}$. By the previous theorem, either $-x_n \to c$ for some $c \in \mathbb{R}$ as $n \to \infty$, or else $-x_n$ diverges to $+\infty$. In the former case, $x_n \to -c$ as $n \to \infty$ (in particular, it is bounded). In the latter case, x_n diverges to $-\infty$ (in particular, it is unbounded).

Corollary Any monotonic sequence converges to a limit if it is bounded, and diverges to $+\infty$ or $-\infty$ otherwise.

Examples

- If $0 < a < 1$ then $a^n \to 0$ as $n \to \infty$.

Since $a < 1$ and $a > 0$, it follows that $a^{n+1} < a^n$ and $a^n > 0$ for all $n \in \mathbb{N}$. Hence the sequence $\{a^n\}$ is strictly decreasing and bounded. Therefore it converges to some $x \in \mathbb{R}$. Since $a^{n+1} = a^n a$ for all n, it follows that $a^{n+1} \to xa$ as $n \to \infty$. However the sequence $\{a^{n+1}\}$ is a subsequence of $\{a^n\}$, hence it converges to the same limit as $\{a^n\}$. Thus $xa = x$, which implies that $x = 0$.

- If $a > 1$ then $a^n \to +\infty$ as $n \to \infty$.

Since $a > 1$, it follows that $a^{n+1} > a^n > 1$ for all $n \in \mathbb{N}$. Hence the sequence $\{a^n\}$ is strictly increasing. Then $\{a^n\}$ either diverges to $+\infty$ or converges to a limit x. In the latter case we argue as above to obtain that $x = 0$. However this contradicts with $a^n > 1$. Thus $\{a^n\}$ diverges to $+\infty$.

Examples

- If $a > 0$ then $\sqrt[n]{a} \to 1$ as $n \to \infty$.

Remark. By definition, $\sqrt[n]{a}$ is a unique positive number r such that $r^n = a$ (for now, we assume it exists).

If $a \geq 1$ then $a^{n+1} \geq a^n \geq 1$ for all $n \in \mathbb{N}$, which implies that $\sqrt[n(n+1)]{a^{n+1}} \geq \sqrt[n(n+1)]{a^n} \geq 1$. Notice that $\sqrt[n(n+1)]{a^{n+1}} = \sqrt[n]{a}$ and $\sqrt[n(n+1)]{a^n} = \sqrt[n+1]{a}$. Hence $\sqrt[n]{a} \geq \sqrt[n+1]{a} \geq 1$ for all n.
Similarly, in the case $0 < a < 1$ we obtain that
$\sqrt[n]{a} < \sqrt[n+1]{a} < 1$ for all n.

In either case, the sequence $\{\sqrt[n]{a}\}$ is monotonic and bounded. Therefore it converges to a limit x. Then the sequence $\{\sqrt[2n]{a}\}$ also converges to x since it is a subsequence of $\{\sqrt[n]{a}\}$. At the same time, $(\sqrt[2n]{a})^2 = \sqrt[n]{a}$, which implies that $x^2 = x$. Hence $x = 0$ or $x = 1$. However the limit cannot be 0 since $\sqrt[n]{a} \geq \min(a, 1) > 0$. Thus $x = 1$.

Examples

- $3,\ 3+\sqrt{3},\ 3+\sqrt{3+\sqrt{3}},\ldots$

That is, $x_1 = 3$ and $x_{n+1} = 3+\sqrt{x_n}$ for all $n \in \mathbb{N}$.

First we show that $\{x_n\}$ is increasing (by induction on n). We have $x_2 = 3+\sqrt{x_1} = 3+\sqrt{3} > 3 = x_1$. Further, if $x_{n+1} > x_n$ for some $n \in \mathbb{N}$, then $x_{n+2} = 3+\sqrt{x_{n+1}} > 3+\sqrt{x_n} = x_{n+1}$.

Next we show that $x_n < 6$ for all $n \in \mathbb{N}$ (also by induction on n). Indeed, $x_1 = 3 < 6$, and if $x_n < 6$ for some $n \in \mathbb{N}$, then $x_{n+1} = 3+\sqrt{x_n} < 3+\sqrt{6} < 3+\sqrt{9} = 6$.

Thus the sequence $\{x_n\}$ is increasing and bounded. Therefore it converges to a limit L. Note that $L > 3 = x_1$. Since $x_{n+1} = 3+\sqrt{x_n}$, it follows that $(x_{n+1}-3)^2 = x_n$. As the sequence $\{x_{n+1}\}$ converges to the same limit L, the limit theorems imply that $(L-3)^2 = L$. Then $L^2 - 7L + 9 = 0$ and $L = (7 \pm \sqrt{13})/2$. Since $L > 3$, we have $L = (7+\sqrt{13})/2$.