MATH 409

Advanced Calculus I

Lecture 11: More examples of limits.

Limit of a sequence

Definition. Sequence $\{x_n\}$ of real numbers is said to **converge** to a real number a if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N$. The number a is called the **limit** of $\{x_n\}$.

A sequence is called **convergent** if it has a limit and **divergent** otherwise.

Properties of convergent sequences:

- the limit is unique;
- any convergent sequence is bounded;
- any subsequence of a convergent sequence converges to the same limit;
- modifying a finite number of elements cannot affect convergence of a sequence or change its limit;
- rearranging elements of a sequence cannot affect its convergence or change its limit.

Limit theorems

Theorem 1 If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$ and $x_n \leq w_n \leq y_n$ for all sufficiently large n, then $\lim_{n\to\infty} w_n = a$.

Theorem 2 If $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$, and $x_n \leq y_n$ for all sufficiently large n, then $a \leq b$.

Theorem 3 If $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$, then $\lim_{n\to\infty}(x_n + y_n) = a + b$, $\lim_{n\to\infty}(x_n - y_n) = a - b$, and $\lim_{n\to\infty} x_n y_n = ab$. If, additionally, $b \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n / y_n = a/b$.

Monotonic sequences

Definition. A sequence $\{x_n\}$ is called **nondecreasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. It is called (strictly) **increasing** if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.

Likewise, the sequence $\{x_n\}$ is called **nonincreasing** if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. It is called (strictly) **decreasing** if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.

All of the above sequences are called **monotonic**.

Theorem Any monotonic sequence converges to a limit if bounded, and diverges to $+\infty$ or $-\infty$ otherwise.

Examples

- $$\lim_{n\to\infty} \frac{\sin(e^n)}{n} = 0.$$

$-1/n \leq \sin(e^n)/n \leq 1/n$ for all $n \in \mathbb{N}$ since $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$. We already know that $1/n \to 0$ as $n \to \infty$. Then $-1/n \to -1 \cdot 0 = 0$ as $n \to \infty$. By the Squeeze Theorem, $\sin(e^n)/n \to 0$ as $n \to \infty$.

- $$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

The sequence $\{1/2^n\}$ is a subsequence of $\{1/n\}$. Hence it is converging to the same limit. Alternatively, we can show by induction that $2^n > n$ for all $n \in \mathbb{N}$. Then $0 < 2^{-n} < n^{-1}$ and we can apply the Squeeze Theorem.

Examples

- The sequence $x_n = \left(1 + \frac{1}{n}\right)^n$, $n = 1, 2, 3, \dots,$

is increasing and bounded, hence it is convergent.

Remark. The limit is the number $e = 2.71828\dots$

First let us show that $\{x_n\}$ is increasing. For any $n \in \mathbb{N}$,

$$x_n = \left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n = \frac{(n+1)^n}{n^n}.$$

If $n \geq 2$ then, similarly, $x_{n-1} = \frac{n^{n-1}}{(n-1)^{n-1}}$. Hence

$$\frac{x_n}{x_{n-1}} = \frac{(n+1)^n}{n^n} \cdot \frac{(n-1)^{n-1}}{n^{n-1}} = \left(\frac{(n+1)(n-1)}{n^2}\right)^{n-1} \cdot \frac{n+1}{n}$$

$$= \left(\frac{n^2-1}{n^2}\right)^{n-1} \cdot \frac{n+1}{n} = \left(1 - \frac{1}{n^2}\right)^{n-1} \left(1 + \frac{1}{n}\right).$$

To proceed, we need the following estimate.

Lemma If $0 < x < 1$, then $(1-x)^k \geq 1 - kx$ for all $k \in \mathbb{N}$.

Using the lemma, we obtain that

$$\frac{x_n}{x_{n-1}} = \left(1 - \frac{1}{n^2}\right)^{n-1} \left(1 + \frac{1}{n}\right) \geq \left(1 - \frac{n-1}{n^2}\right)\left(1 + \frac{1}{n}\right)$$

$$= 1 - \frac{n-1}{n^2} + \frac{1}{n} - \frac{n-1}{n^3} = 1 + \frac{1}{n^2} - \frac{n-1}{n^3} = 1 + \frac{1}{n^3} > 1.$$

Thus the sequence $\{x_n\}$ is strictly increasing.

Proof of the lemma: The lemma is proved by induction on k. The case $k = 1$ is trivial as $(1-x)^1 = 1 - 1 \cdot x$. Now assume that the inequality $(1-x)^k \geq 1 - kx$ holds for some $k \in \mathbb{N}$ and all $x \in (0,1)$. Then $(1-x)^{k+1} = (1-x)^k(1-x)$ $\geq (1-kx)(1-x) = 1 - kx - x + kx^2 > 1 - (k+1)x$. ■

Remark. According to the Binomial Formula,

$$(1-x)^k = 1 - kx + \tfrac{k(k-1)}{2}x^2 - \dots$$

Now let us show that the sequence $\{x_n\}$ is bounded. Since $\{x_n\}$ is increasing, it is enough to show that it is bounded above. By the Binomial Formula,

$$x_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \left(\frac{1}{n}\right)^k.$$

Observe that $\dfrac{n!}{(n-k)!} \left(\dfrac{1}{n}\right)^k = \dfrac{n(n-1)\dots(n-k+1)}{n^k} \leq 1.$

It follows that $x_n \leq \sum_{k=0}^{n} \dfrac{1}{k!} = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}.$

Further observe that $k! \geq 2^{k-1}$ for all $k \geq 0$. Therefore we obtain

$$x_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 3 - \frac{1}{2^{n-1}} < 3.$$

Examples

- $\sqrt[n]{n} \to 1$ as $n \to \infty$.

The sequence $x_n = \sqrt[n]{n}$, $n \in \mathbb{N}$ is eventually decreasing. We have $1 < \sqrt{2} < \sqrt[3]{3}$, but then $\sqrt[3]{3} > \sqrt[4]{4} > \sqrt[5]{5} > \dots$ That is, $x_{n+1} < x_n$ for all $n \geq 3$. Indeed, using the estimate from the previous example, we obtain

$$\left(\frac{x_{n+1}}{x_n}\right)^{n(n+1)} = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\left(1 + \frac{1}{n}\right)^n < \frac{3}{n} \leq 1.$$

Since the sequence $\{\sqrt[n]{n}\}$ is eventually decreasing and bounded below (by 1), it converges to a limit L. Then the sequence $\{\sqrt[2n]{2n}\}$ also converges to L since it is a subsequence of $\{\sqrt[n]{n}\}$. At the same time, $(\sqrt[2n]{2n})^2 = \sqrt[n]{2n} = \sqrt[n]{2}\,\sqrt[n]{n}$. We already know that $\sqrt[n]{2} \to 1$ as $n \to \infty$. It follows that $L^2 = L$. Hence $L = 0$ or $L = 1$. However the limit cannot be 0 since 1 is a lower bound. Thus $L = 1$.