MATH 409

Advanced Calculus I

Lecture 12:
Bolzano-Weierstrass theorem.
Cauchy sequences.

Nested intervals property

Definition. A sequence of sets $I_1, I_2, \dots$ is called **nested** if $I_1 \supset I_2 \supset \dots$, that is, $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$.

Theorem If $\{I_n\}$ is a nested sequence of nonempty closed bounded intervals, then the intersection $\bigcap_{n\in\mathbb{N}} I_n$ is nonempty. Moreover, if lengths $|I_n|$ of the intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then the intersection consists of a single point.

Remark 1. The theorem may not hold if the intervals $I_1, I_2, \dots$ are open. Counterexample: $I_n = (0, 1/n)$, $n \in \mathbb{N}$. The intervals are nested and bounded, but their intersection is empty since $1/n \to 0$ as $n \to \infty$.

Remark 2. The theorem may not hold if the intervals $I_1, I_2, \dots$ are not bounded. Counterexample: $I_n = [n, \infty)$, $n \in \mathbb{N}$. The intervals are nested and closed, but their intersection is empty since the sequence $\{n\}$ diverges to $+\infty$.

Proof of the theorem

Let $I_n = [a_n, b_n]$, $n = 1, 2, \dots$ Since the sequence $\{I_n\}$ is nested, it follows that the sequence $\{a_n\}$ is nondecreasing while $\{b_n\}$ is nonincreasing. Besides, both sequences are bounded (since both are contained in the interval I_1). Hence both are convergent: $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Since $a_n \le b_n$ for all $n \in \mathbb{N}$, the Comparison Theorem implies that $a \le b$. We claim that $\bigcap_{n \in \mathbb{N}} I_n = [a, b]$. Indeed, we have $a_n \le a$ for all $n \in \mathbb{N}$ (by the Comparison Theorem applied to $a_1, a_2, \dots$ and the constant sequence $a_n, a_n, a_n \dots$). Similarly, $b \le b_n$ for all $n \in \mathbb{N}$. Therefore $[a, b]$ is contained in the intersection. On the other hand, if $x < a$ then $x < a_n$ for some n so that $x \notin I_n$. Similarly, if $x > b$ then $x > b_m$ for some m so that $x \notin I_m$. This proves the claim.

Clearly, the length of $[a, b]$ cannot exceed $|I_n|$ for any $n \in \mathbb{N}$. Therefore $|I_n| \to 0$ as $n \to \infty$ implies that $[a, b]$ is a degenerate interval: $a = b$.

Bolzano-Weierstrass Theorem

Theorem Every bounded sequence of real numbers has a convergent subsequence.

Proof: Let $\{x_n\}$ be a bounded sequence of real numbers. We are going to build a nested sequence of intervals $I_n = [a_n, b_n]$, $n = 1, 2, \dots$, such that each I_n contains infinitely many elements of $\{x_n\}$ and $|I_{n+1}| = |I_n|/2$ for all $n \in \mathbb{N}$. The sequence is built inductively. First we set I_1 to be any closed bounded interval that contains all elements of $\{x_n\}$ (such an interval exists since the sequence $\{x_n\}$ is bounded). Now assume that for some $n \in \mathbb{N}$ the interval I_n is already chosen and it contains infinitely many elements of the sequence $\{x_n\}$. Then at least one of the subintervals $I' = [a_n, (a_n + b_n)/2)]$ and $I'' = [(a_n + b_n)/2, b_n]$ also contains infinitely many elements of $\{x_n\}$. We set I_{n+1} to be such a subinterval. By construction, $I_{n+1} \subset I_n$ and $|I_{n+1}| = |I_n|/2$.

Proof (continued): Since $|I_{n+1}| = |I_n|/2$ for all $n \in \mathbb{N}$, it follows by induction that $|I_n| = |I_1|/2^{n-1}$ for all $n \in \mathbb{N}$. As a consequence, $|I_n| \to 0$ as $n \to \infty$. By the nested intervals property, the intersection of the intervals $I_1, I_2, I_3, \dots$ consists of a single number a.

Next we are going to build a strictly increasing sequence of natural numbers $n_1, n_2, \dots$ such that $x_{n_k} \in I_k$ for all $k \in \mathbb{N}$. The sequence is built inductively. First we choose n_1 so that $x_{n_1} \in I_1$. Now assume that for some $k \in \mathbb{N}$ the number n_k is already chosen. Since the interval I_{k+1} contains infinitely many elements of the sequence $\{x_n\}$, there exists $m > n_k$ such that $x_m \in I_{k+1}$. We set $n_{k+1} = m$.

Now we claim that the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of the sequence $\{x_n\}$ converges to a. Indeed, for any $k \in \mathbb{N}$ the points x_{n_k} and a both belong to the interval I_k. Hence $|x_{n_k} - a| \leq |I_k|$. Since $|I_k| \to 0$ as $k \to \infty$, it follows that $x_{n_k} \to a$ as $k \to \infty$.

Theorem Any sequence of real numbers has a monotonic subsequence.

Proof: Let $\{x_n\}$ be a sequence of real numbers. We call a natural number m a **turn-back index** for this sequence if $x_m \geq x_n$ for all $n > m$. Let us consider two possible cases.

Case 1: there are infinitely many turn-back indices. Let $m_1, m_2, m_3, \ldots$ be the list of all turn-back indices arranged in ascending order. Then $x_{m_1}, x_{m_2}, x_{m_3}, \ldots$ is a subsequence of $\{x_n\}$. By definition of a turn-back index, this subsequence is nonincreasing.

Case 2: there are only finitely many turn-back indices. In this case, there exists $M \in \mathbb{N}$ that is greater than any turn-back index. Now we can build inductively an increasing sequence of natural numbers $m_1 < m_2 < m_3 < \ldots$ such that $m_1 = M$ and $x_{m_{n+1}} > x_{m_n}$ for all $n \geq 1$ (the choice of m_{n+1} is always possible since m_n is not a turn-back index). By construction, $\{x_{m_n}\}$ is an increasing subsequence of the original sequence.

Cauchy sequences

Definition. A sequence $\{x_n\}$ of real numbers is called a **Cauchy sequence** if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ whenever $n, m \geq N$.

Theorem Any convergent sequence is Cauchy.

Proof: Let $\{x_n\}$ be a convergent sequence and a be its limit. Then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon/2$ whenever $n \geq N$. Now for any natural numbers $n, m \geq N$ we have

$$|x_n - x_m| = |x_n - a + a - x_m| \leq |x_n - a| + |x_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence.

Theorem Any Cauchy sequence is convergent.

Proof: Suppose $\{x_n\}$ is a Cauchy sequence. First let us show that this sequence is bounded. Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < 1$ whenever $n, m \geq N$. In particular, $|x_n - x_N| < 1$ for all $n \geq N$. Then $|x_n| = |(x_n - x_N) + x_N| \leq |x_n - x_N| + |x_N| < |x_N| + 1$. It follows that for any $n \in \mathbb{N}$ we have $|x_n| \leq M$, where $M = \max(|x_1|, |x_2|, \dots, |x_{N-1}|, |x_N| + 1)$.

Now the Bolzano-Weierstrass theorem implies that $\{x_n\}$ has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converging to some $a \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $K_\varepsilon \in \mathbb{N}$ such that $|x_{n_k} - a| < \varepsilon/2$ for all $k \geq K_\varepsilon$. Also, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon/2$ whenever $n, m \geq N_\varepsilon$. Let $k = \max(K_\varepsilon, N_\varepsilon)$. Then $k \geq K_\varepsilon$ and $n_k \geq k \geq N_\varepsilon$. Therefore for any $n \geq N_\varepsilon$ we obtain $|x_n - a| = |(x_n - x_{n_k}) + (x_{n_k} - a)| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus the entire sequence $\{x_n\}$ converges to a.