MATH 409

Advanced Calculus I

Lecture 13:
Limit points.
Upper and lower limits.

Limit points

Definition. A **limit point** (or **subsequential limit** or **cluster point**) of a sequence $\{x_n\}$ is the limit of any convergent subsequence of $\{x_n\}$.

Recall that the ε-**neighborhood** of a point $a \in \mathbb{R}$ is the interval $(a - \varepsilon, a + \varepsilon)$. For any finite collection of distinct points $a_1, a_2, \dots, a_k \in \mathbb{R}$ there exists $\varepsilon > 0$ such that the ε-neighborhoods of all those points are disjoint.

Theorem 1 A point $a \in \mathbb{R}$ is the limit of a sequence $\{x_n\}$ of real numbers if and only if any ε-neighborhood of a contains *all but finitely many* terms of the sequence (counting with repetitions).

Theorem 2 A point $a \in \mathbb{R}$ is a limit point of a sequence $\{x_n\}$ of real numbers if and only if any ε-neighborhood of a contains *infinitely many* terms of the sequence (counting with repetitions).

Limit points

Examples and properties:

- A convergent sequence has only one limit point, its limit.
- Any bounded sequence has at least one limit point.
- If a bounded sequence is not convergent, then it has at least two limit points.
- The sequence $1, -1, 1, -1, 1, -1, \ldots$ has two limit points, 1 and -1.
- If all elements of a sequence belong to a closed interval $[a, b]$, then all its limit points belong to $[a, b]$ as well.
- The sequence $\{x_n\}$, where $x_n = \{n\sqrt{2}\}$ (fractional part of $n\sqrt{2}$) fills densely the interval $(0, 1)$ (due to Jacobi's Theorem). Hence the set of its limit points is $[0, 1]$.
- The set of limit points of the sequence $\{\sin n\}$ is the entire interval $[-1, 1]$.

Extended real line

The **extended real line** $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is obtained by adding to $\mathbb{R}$ two extra points "at infinity". By definition, the ε-neighborhood of $+\infty$ is $(1/\varepsilon, \infty)$ and the ε-neighborhood of $-\infty$ is $(-\infty, -1/\varepsilon)$. Then divergence to $+\infty$ or $-\infty$ can be understood as convergence in $\overline{\mathbb{R}}$.
In particular, $+\infty$ and $-\infty$ can be limit points now.

More examples and properties:

• If a sequence diverges to $+\infty$ or $-\infty$, then this is its only limit point in $\overline{\mathbb{R}}$.

• Any sequence has at least one limit point in $\overline{\mathbb{R}}$.

• A sequence $\{x_n\}$ has no finite limit points if and only if $|x_n| \to \infty$ as $n \to \infty$.

• If a sequence has only finitely many limit points, then it can be decomposed into finitely many convergent (in $\overline{\mathbb{R}}$) subsequences.

Upper and lower limits

Let $\{x_n\}$ be a bounded sequence of real numbers. For any $n \in \mathbb{N}$ let E_n denote the set of all numbers of the form x_k, where $k \geq n$. The set E_n is bounded, hence $\sup E_n$ and $\inf E_n$ exist. Observe that the sequence $\{\sup E_n\}$ is nonincreasing, the sequence $\{\inf E_n\}$ is nondecreasing (since $E_1, E_2, \dots$ are nested sets), and both are bounded. Therefore both sequences are convergent.

Definition. The limit of $\{\sup E_n\}$ is called the **limit supremum** of the sequence $\{x_n\}$ and denoted $\limsup\limits_{n\to\infty} x_n$.

The limit of $\{\inf E_n\}$ is called the **limit infimum** of the sequence $\{x_n\}$ and denoted $\liminf\limits_{n\to\infty} x_n$.

Properties of lim sup *and* lim inf*.*

- $\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n$.

- If $C > \limsup_{n\to\infty} x_n$, then C is an **eventual upper bound** for $\{x_n\}$, which means that $x_n \le C$ when n large enough. If $C < \limsup_{n\to\infty} x_n$, then C is not an eventual upper bound.

- $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ are limit points of $\{x_n\}$.

- All limit points of $\{x_n\}$ are contained in the interval $\left[\liminf_{n\to\infty} x_n, \limsup_{n\to\infty} x_n\right]$.

- The sequence $\{x_n\}$ converges to a limit a if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = a$.