MATH 409

Advanced Calculus I

Lecture 14: Convergence of infinite series.

Infinite series

Definition. Given a sequence $\{a_n\}$ of real numbers, an expression $a_1 + a_2 + \cdots + a_n + \cdots$ or $\sum_{n=1}^{\infty} a_n$ is called an **infinite series** with **terms** a_n. The **partial sum** of order n of the series is defined by $s_n = a_1 + a_2 + \cdots + a_n$. If the sequence $\{s_n\}$ converges to a limit $s \in \mathbb{R}$, we say that the series **converges** to s or that s is the **sum** of the series and write $\sum_{n=1}^{\infty} a_n = s$. Otherwise the series **diverges**.

Theorem (Cauchy Criterion) An infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $|a_n + a_{n+1} + \cdots + a_m| < \varepsilon$.

Proof: Let $\{s_n\}$ be the sequence of partial sums. Then $a_n + a_{n+1} + \cdots + a_m = s_m - s_{n-1}$. Consequently, the condition of the theorem is equivalent to the condition that $\{s_n\}$ be a Cauchy sequence. As we know, a sequence is convergent if and only if it is a Cauchy sequence.

Examples

- $$\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}+\cdots=1.$$

The partial sums s_n of this series satisfy $s_n = 1 - 2^{-n}$ for all $n \in \mathbb{N}$. Thus $s_n \to 1$ as $n \to \infty$.

- $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}+\cdots=1.$$

Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, the partial sums s_n of this series satisfy $s_n = 1 - \frac{1}{n+1}$. Thus $s_n \to 1$ as $n \to \infty$.

- $\sum_{n=1}^{\infty}(-1)^n = -1+1-1+\dots$ diverges.

The partial sums s_n satisfy $s_n = -1$ for odd n and $s_n = 0$ for even n. Hence the sequence $\{s_n\}$ has no limit.

Some properties of infinite series

Theorem (Trivial Test) If the terms of an infinite series do not converge to zero, then the series diverges.

Theorem If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

and

$$\sum_{n=1}^{\infty}(ra_n) = r\sum_{n=1}^{\infty} a_n$$

for any $r \in \mathbb{R}$.

Theorem If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

Example

- The **geometric series** $\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$, in which case its sum is $\dfrac{1}{1-x}$.

In the case $|x| \geq 1$, the series fails the Trivial Test. For any $x \neq 1$, the partial sums s_n of the geometric series satisfy

$$s_n = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

In the case $|x| < 1$, we obtain that $s_n \to 1/(1-x)$ as $n \to \infty$.

Series with nonnegative terms

Suppose that a series $\sum_{n=1}^{\infty} a_n$ has nonnegative terms, $a_n \geq 0$ for all $n \in \mathbb{N}$. Then the sequence of partial sums $s_n = a_1 + a_2 + \cdots + a_n$ is nondecreasing. It follows that $\{s_n\}$ converges to a finite limit if bounded and diverges to $+\infty$ otherwise. In the latter case, we write $\sum_{n=1}^{\infty} a_n = \infty$.

Theorem (Direct Comparison Test) Suppose that $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \leq b_n$ for large n. Then convergence of the series $\sum_{n=1}^{\infty} b_n$ implies convergence of $\sum_{n=1}^{\infty} a_n$ while $\sum_{n=1}^{\infty} a_n = \infty$ implies $\sum_{n=1}^{\infty} b_n = \infty$.

Proof: Since changing a finite number of terms does not affect convergence of a series, it is no loss to assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then the partial sums $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ satisfy $s_n \leq t_n$ for all n. Consequently, if $s_n \to +\infty$ as $n \to \infty$, then also $t_n \to +\infty$ as $n \to \infty$. Conversely, if $\{t_n\}$ is bounded, then so is $\{s_n\}$.

Examples

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Indeed, $0 < \frac{1}{n^2} < \frac{1}{n(n-1)}$ for all $n \geq 2$. Since the series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is convergent, it remains to apply the Direct Comparison Test.

- $\sum_{n=1}^{\infty} e^{-n^2}$ converges.

We have $0 < e^{-n^2} \leq e^{-n}$ for all $n \in \mathbb{N}$. The geometric series $\sum_{n=1}^{\infty} e^{-n}$ is convergent since $0 < e^{-1} < 1$. By the Direct Comparison Test, $\sum_{n=1}^{\infty} e^{-n^2}$ is convergent as well.