MATH 409

Advanced Calculus I

Lecture 15:
Tests for convergence.

Convergence of infinite series

[Cauchy Criterion] An infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $|a_n + a_{n+1} + \cdots + a_m| < \varepsilon$.

[Trivial Test] If the terms of an infinite series do not converge to zero, then the series diverges.

[Series with nonnegative terms] A series $\sum_{n=1}^{\infty} a_n$ with nonnegative terms converges if and only if the sequence of partial sums $s_n = a_1 + a_2 + \cdots + a_n$ is bounded above.

[Direct Comparison Test] Suppose that $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \leq b_n$ for large n. Then convergence of the series $\sum_{n=1}^{\infty} b_n$ implies convergence of $\sum_{n=1}^{\infty} a_n$ while $\sum_{n=1}^{\infty} a_n = \infty$ implies $\sum_{n=1}^{\infty} b_n = \infty$.

Harmonic series

- $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \ldots \quad \text{diverges.}$$

Indeed,

$$\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}, \quad \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}, \ \ldots$$

In general, $\displaystyle\sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n} > \frac{2^k}{2^{k+1}} = \frac{1}{2}.$

It follows that $\displaystyle\sum_{n=1}^{2^k} \frac{1}{n} > 1 + \frac{k}{2}.$

Thus the sequence of partial sums is unbounded.

Cauchy's Condensation Test

Theorem Suppose $\{a_n\}$ is a nonincreasing sequence of positive numbers. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the "condensed" series $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Proof: Since $\{a_n\}$ is a nonincreasing sequence, for any $i \in \mathbb{N}$ we have $2^i a_{2^{i+1}} \leq a_{2^i+1} + a_{2^i+2} + a_{2^i+3} + \cdots + a_{2^{i+1}} \leq 2^i a_{2^i}$. Summing this up over i from 1 to k, we obtain

$$\sum_{i=1}^{k} 2^i a_{2^{i+1}} \leq \sum_{n=3}^{2^{k+1}} a_n \leq \sum_{i=1}^{k} 2^i a_{2^i}.$$

Let $\{s_n\}$ be the partial sums of the original series and $\{t_k\}$ be the partial sums of the condensed one. The latter inequalities are equivalent to $\frac{1}{2}(t_{k+1} - t_1) \leq s_{2^{k+1}} - s_2 \leq t_k$. It follows that the sequence $\{t_k\}$ is bounded if and only if the sequence $\{s_{2^k}\}$ is bounded. Since the sequence $\{s_n\}$ is nondecreasing, it is bounded if and only if its subsequence $\{s_{2^k}\}$ is bounded.

Examples

- $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \ldots, \;\; p > 0.$

Since $a_n = 1/n^p$, the condensed series is

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \frac{2^k}{(2^k)^p} = \sum_{k=1}^{\infty} \left(\frac{2}{2^p}\right)^k.$$

This is a geometric series. It converges if $2/2^p < 1$, i.e. $p > 1$, and diverges if $2/2^p \geq 1$, i.e. $p \leq 1$.

- $\sum_{n=2}^{\infty} \frac{1}{n \log^p n}, \;\; p > 0.$

Since $a_n = (n \log^p n)^{-1}$, the condensed series is

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \frac{2^k}{2^k (\log(2^k))^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \frac{1}{\log^p 2} \sum_{k=1}^{\infty} \frac{1}{k^p}.$$

By the above it converges if $p > 1$ and diverges if $p \leq 1$.

d'Alembert's Ratio Test

Theorem Let $\{a_n\}$ be a sequence of positive numbers. Suppose that the limit

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists (finite or infinite).

(i) If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. In the case $r = 1$, the Ratio Test is inconclusive. For example, consider a series $\sum_{n=1}^{\infty} n^{-p}$, where $p > 0$. Then

$$r = \lim_{n\to\infty} \frac{(n+1)^{-p}}{n^{-p}} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^{p} = 1$$

for all $p > 0$. However the series converges for $p > 1$ and diverges for $0 < p \leq 1$.

Theorem Let $\{a_n\}$ be a sequence of positive numbers. Suppose that the limit $r = \lim_{n\to\infty} a_{n+1}/a_n$ exists (finite or infinite).

(i) If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: If $r > 1$, then $a_{n+1}/a_n > 1$ for n large enough. It follows that the sequence $\{a_n\}$ is eventually increasing. Then $a_n \not\to 0$ as $n \to \infty$ so that the series $\sum_{n=1}^{\infty} a_n$ diverges due to the Trivial Test.

In the case $r < 1$, choose some $x \in (r, 1)$. Then $a_{n+1}/a_n < x$ for n large enough. Consequently, $a_{n+1}/x^{n+1} < a_n/x^n$ for n large enough. That is, the sequence $\{a_n/x^n\}$ is eventually decreasing. It follows that this sequence is bounded. Hence $a_n \leq Cx^n$ for some $C > 0$ and all $n \in \mathbb{N}$. Since $0 < r < x < 1$, the geometric series $\sum_{n=1}^{\infty} x^n$ converges. So does the series $\sum_{n=1}^{\infty} Cx^n$. By the Direct Comparison Test, the series $\sum_{n=1}^{\infty} a_n$ converges as well.

Cauchy's Root Test

Theorem Let $\{a_n\}$ be a sequence of positive numbers and

$$r = \limsup_{n\to\infty} \sqrt[n]{a_n}.$$

(i) If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
(ii) If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: If $r > 1$, then $\sup_{k\geq n} \sqrt[k]{a_k} \geq r > 1$ for all $n \in \mathbb{N}$. Therefore for any $n \in \mathbb{N}$ there exists $k(n) \geq n$ such that $a_{k(n)}^{1/k(n)} > 1$. In particular, $a_{k(n)} > 1$. It follows that $a_k \not\to 0$ as $k \to \infty$ so that the series $\sum_{k=1}^{\infty} a_k$ diverges due to the Trivial Test.

In the case $r < 1$, choose some $x \in (r, 1)$. Then $\sup_{k\geq n} \sqrt[k]{a_k} < x$ for some $n \in \mathbb{N}$. This implies that $a_k < x^k$ for all $k \geq n$. Since $0 < r < x < 1$, the geometric series $\sum_{k=1}^{\infty} x^k$ converges. By the Direct Comparison Test, the series $\sum_{k=1}^{\infty} a_k$ converges as well.

Examples

- $\sum_{n=1}^{\infty} \dfrac{n}{2^n} = \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \dfrac{4}{16} + \dfrac{5}{32} + \dots$

If $a_n = n/2^n$, then

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \left(\frac{n}{2^n}\right)^{-1} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} \to \frac{1}{2}$$

as $n \to \infty$. By the Ratio Test, the series converges.

- $\sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots, \quad x > 0.$

Let $a_n = x^n/n!$, $n \in \mathbb{N}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \left(\frac{x^n}{n!}\right)^{-1} = \frac{x}{n+1} \to 0 \text{ as } n \to \infty.$$

By the Ratio Test, the series converges for all $x > 0$.

Examples

- $$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} = \frac{(1!)^2}{2!} + \frac{(2!)^2}{4!} + \frac{(3!)^2}{6!} + \dots$$

If $a_n = (n!)^2/(2n)!$, then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n+1}{2(2n+1)} = \frac{1+n^{-1}}{4+2n^{-1}} \to \frac{1}{4}$$

as $n \to \infty$. By the Ratio Test, the series converges.

- $$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} = \frac{1}{2} + \left(\frac{2}{3}\right)^4 + \left(\frac{3}{4}\right)^9 + \dots$$

If $a_n = (n/(n+1))^{n^2}$, then

$$\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left(1+\frac{1}{n}\right)^{-n} \to \frac{1}{e}$$

as $n \to \infty$. By the Root Test, the series converges.