MATH 409

Advanced Calculus I

Lecture 16:
Tests for convergence (continued).
Alternating series.

Tests for convergence

[Condensation Test] Suppose $\{a_n\}$ is a nonincreasing sequence of positive numbers. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the "condensed" series $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

[Ratio Test] Let $\{a_n\}$ be a sequence of positive numbers. Suppose that the limit $r = \lim_{n\to\infty} a_{n+1}/a_n$ exists (finite or infinite). **(i)** If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges. **(ii)** If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

[Root Test] Let $\{a_n\}$ be a sequence of positive numbers and

$$r = \limsup_{n\to\infty} \sqrt[n]{a_n}.$$

(i) If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
(ii) If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Refined ratio tests

[Raabe's Test] Let $\{a_n\}$ be a sequence of positive numbers. Suppose that the limit $L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right)$ exists (finite or infinite). Then the series $\sum_{n=1}^{\infty} a_n$ converges if $L > 1$, and diverges if $L < 1$.

Remark. If $-\infty < L < \infty$ then $a_{n+1}/a_n \to 1$ as $n \to \infty$.

[Gauss's Test] Let $\{a_n\}$ be a sequence of positive numbers. Suppose that $\frac{a_n}{a_{n+1}} = 1 + \frac{L}{n} + \frac{\gamma_n}{n^{1+\varepsilon}}$, where L is a constant, $\varepsilon > 0$, and $\{\gamma_n\}$ is a bounded sequence. Then the series $\sum_{n=1}^{\infty} a_n$ converges if $L > 1$, and diverges if $L \le 1$.

Remark. Under the assumptions of the Gauss Test, $n(a_n/a_{n+1} - 1) \to L$ as $n \to \infty$.

Example

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

The usual Ratio Test is inconclusive. Indeed, $a_n = n^{-p}$ and

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{-p}}{n^{-p}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-p} = 1$$

for all $p > 0$. On the other hand,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\Big((1 + 1/n)^p - 1\Big) = \frac{(1 + 1/n)^p - 1}{1/n},$$

which converges to p as $n \to \infty$, the derivative of the function $f(x) = x^p$ at $x = 1$. Hence the Raabe Test is conclusive for $p \neq 1$. Further,

$$\frac{a_n}{a_{n+1}} = \left(1 + \frac{1}{n}\right)^p = 1 + \frac{p}{n} + \frac{\gamma_n}{n^2},$$

where $\{\gamma_n\}$ is bounded. Hence the Gauss Test is conclusive for all $p > 0$.

Integral test

Theorem Suppose that a function $f : [1, \infty) \to \mathbb{R}$ is positive and nonincreasing on $[1, \infty)$. Then
(i) a sequence $\{y_n\}$ is bounded and nonincreasing, where

$$y_n = f(1) + f(2) + \cdots + f(n) - \int_1^n f(x)\,dx, \quad n = 1, 2, \ldots$$

(ii) the series $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if the function f is improperly integrable on $[1, \infty)$.

Remark. Suppose $F(x) = \int f(x)\,dx$ is the antiderivative. Then the improper integral $\int_1^{\infty} f(x)\,dx$ converges if and only if $F(x)$ converges to a finite limit as $x \to +\infty$.

Idea of the proof: $f(n+1) \le \int_n^{n+1} f(x)\,dx \le f(n)$.

Examples

- The harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

Indeed, $\int_1^n x^{-1}\,dx = \log n \to +\infty$ as $n \to \infty$. By the Integral Test, the series is divergent. Moreover, the sequence $y_n = \sum_{k=1}^n k^{-1} - \log n$ is bounded and decreasing (and hence convergent).

- $\sum_{n=2}^{\infty} \dfrac{1}{n \log^2 n}$ converges.

The antiderivative of $f(x) = (x \log^2 x)^{-1}$ on $(1, \infty)$ is

$$\int \frac{dx}{x \log^2 x} = \int \frac{d(\log x)}{\log^2 x} = -\frac{1}{\log x} + C.$$

Since the antiderivative converges to a finite limit at $+\infty$, the function f is improperly integrable on $[2, \infty)$. By the Integral Test, the series converges.

Alternating Series Test

Definition. An infinite series $\sum_{n=1}^{\infty} a_n$ is called **alternating** if any two neighboring terms have different signs: $a_n a_{n+1} < 0$ for all $n \in \mathbb{N}$.

Theorem (Leibniz Criterion) If $\{a_n\}$ is a nonincreasing sequence of positive numbers and $a_n \to 0$ as $n \to \infty$, then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

converges.

Theorem (Leibniz Criterion) If $\{a_n\}$ is a nonincreasing sequence of positive numbers and $a_n \to 0$ as $n \to \infty$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$ converges.

Proof: Let $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ be the partial sum of order n of the series. For any $n \in \mathbb{N}$ we have

$$s_{2n} = s_{2n-1} - a_{2n} < s_{2n-1}.$$

Since the sequence $\{a_n\}$ is nonincreasing, we also have

$$s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} \le s_{2n-1},$$
$$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \ge s_{2n}.$$

Hence $s_{2n} \le s_{2n+2} < s_{2n+1} \le s_{2n-1}$ for all $n \in \mathbb{N}$. It follows that a subsequence $\{s_{2n}\}$ is nondecreasing, a subsequence $\{s_{2n-1}\}$ is nonincreasing, and both are bounded. Hence both subsequences are convergent. Since $s_{2n-1} - s_{2n} = a_{2n} \to 0$ as $n \to \infty$, both subsequences converge to the same limit L. Then L is the limit of the entire sequence $\{s_n\}$.

Examples

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

  The series converges due to the Alternating Series Test. One can show that the sum is $\log 2$.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2n-1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots$

  After multiplying all terms by -1, the series satisfy all conditions of the Alternating Series Test. It follows that the series converges. One can show that the sum is $-\pi/4$.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n}{2n-1} = 1 - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \dots$

  The series is alternating and the terms decrease in absolute value. However the absolute values of terms converge to $1/2$ instead of 0. Hence the series diverges.