MATH 409

Advanced Calculus I

Lecture 17:
Summation by parts.
Absolute convergence of series.

More tests for convergence

[Dirichlet's Test] Suppose $\sum_{n=1}^{\infty} a_n$ is a series with bounded partial sums and $\{b_n\}$ is a monotonic sequence that converges to 0. Then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

[Abel's Test] Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series and $\{b_n\}$ is a bounded monotonic sequence. Then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

The proof of these tests utilizes a technique called **summation by parts**.

Proof: We are going to use the Cauchy Criterion. For any $n, m \in \mathbb{N}$, $n \leq m$, the sum $a_n b_n + a_{n+1} b_{n+1} + \cdots + a_m b_m$ is rewritten as

$$= a_n(b_n - b_{n+1}) + (a_n + a_{n+1})(b_{n+1} - b_{n+2}) + \dots \\ + (a_n + \cdots + a_{m-1})(b_{m-1} - b_m) + (a_n + \cdots + a_m) b_m$$

$$= (s_n - s_{n-1})(b_n - b_{n+1}) + (s_{n+1} - s_{n-1})(b_{n+1} - b_{n+2}) + \dots \\ + (s_{m-1} - s_{n-1})(b_{m-1} - b_m) + (s_m - s_{n-1}) b_m,$$

where $\{s_k\}$ are partial sums of the series $\sum_{k=1}^{\infty} a_k$.

It follows that $|a_n b_n + \cdots + a_m b_m| \leq A_{n,m} B_{n,m}$, where $A_{n,m} = \max_{n \leq i \leq m} |s_i - s_{n-1}|$ and $B_{n,m} = |b_n - b_{n+1}| + |b_{n+1} - b_{n+2}| + \cdots + |b_{m-1} - b_m| + |b_m|$. Since $\{b_k\}$ is monotonic, we have $B_{n,m} = |b_n - b_m| + |b_m|$.

Under the assumptions of the Dirichlet Test, $A_{n,m}$ is bounded while $B_{n,m}$ gets arbitrarily small as $n \to \infty$. Under the assumptions of the Abel Test, $B_{n,m}$ is bounded while $A_{n,m}$ gets arbitrarily small as $n \to \infty$. In either case, $A_{n,m} B_{n,m}$ gets arbitrarily small as $n \to \infty$.

Examples

- $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dots$

  The series converges due to the Dirichlet Test since the series $1 - 1 + 1 - 1 + \dots$ has bounded partial sums.

- $\sum_{n=1}^{\infty} \dfrac{\sin n}{n} = \sin 1 + \dfrac{\sin 2}{2} + \dfrac{\sin 3}{3} + \dfrac{\sin 4}{4} + \dots$

  One can show that the series $\sum_{n=1}^{\infty} \sin n$ has bounded partial sums. Hence the Dirichlet Test applies.

- $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}(n+1)^n}{n^{n+1}} = \dfrac{2^1}{1^2} - \dfrac{3^2}{2^3} + \dfrac{4^3}{3^4} - \dots$

  The series converges due to the Abel Test, with $a_n = (-1)^{n+1}/n$ and $b_n = (n+1)^n/n^n = (1+1/n)^n$.

Absolute convergence

Definition. An infinite series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if $\sum_{n=1}^{\infty} |a_n| < \infty$.

Theorem Any absolutely convergent series is convergent.

Proof: Suppose that a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, i.e., the series $\sum_{n=1}^{\infty} |a_n|$ converges. By the Cauchy Criterion, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\big||a_n| + |a_{n+1}| + \cdots + |a_m|\big| < \varepsilon$$

for $m \geq n \geq N$. Then

$$|a_n + a_{n+1} + \cdots + a_m| \leq |a_n| + |a_{n+1}| + \cdots + |a_m| < \varepsilon$$

for $m \geq n \geq N$. According to the Cauchy Criterion, the series $\sum_{n=1}^{\infty} a_n$ converges.

Examples

- $\sum_{n=1}^{\infty} \dfrac{1}{n^3} = 1 + \dfrac{1}{2^3} + \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dots$

The series converges due to a number of tests. Since it has positive terms, it is absolutely convergent as well.

- $\sum_{n=1}^{\infty} \dfrac{\sin n}{n^2} = \sin 1 + \dfrac{\sin 2}{4} + \dfrac{\sin 3}{9} + \dfrac{\sin 4}{16} + \dots$

Since $|\sin(n)/n^2| \leq 1/n^2$ and the series $\sum_{n=1}^{\infty} 1/n^2$ converges, this series converges absolutely due to the Direct Comparison Test.

- $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dots$

The series converges (due to the Alternating Series Test), but not absolutely as the series $\sum_{n=1}^{\infty} 1/n$ diverges.

Rearrangements

Let $\sum_{n=1}^{\infty} a_n$ be a series and suppose $\sigma : \mathbb{N} \to \mathbb{N}$ is an invertible transformation. Then the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is called a **rearrangement** of the series $\sum_{n=1}^{\infty} a_n$.

Theorem (Dirichlet) If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement $\sum_{n=1}^{\infty} a_{\sigma(n)}$ also converges absolutely and, moreover, to the same sum.

Idea of the proof: Any partial sum of $\sum_{n=1}^{\infty} |a_{\sigma(n)}|$ is bounded above by a partial sum of $\sum_{n=1}^{\infty} |a_n|$, and vice versa.

Theorem (Riemann) If the series $\sum_{n=1}^{\infty} a_n$ converges but not absolutely, then for any $\alpha \in \mathbb{R}$ there is a rearrangement $\sum_{n=1}^{\infty} a_{\sigma(n)}$ that converges to the sum α.

Idea of the proof: Let $\{a_{n_k}\}$ be the subsequence of all positive terms and $\{a_{m_k}\}$ be the subsequence of all the other terms. Then the series $\sum_{k=1}^{\infty} a_{n_k}$ and $\sum_{k=1}^{\infty} a_{m_k}$ both diverge.