MATH 409

Advanced Calculus I

Lecture 18:
Review for Test 1.

Topics for Test 1

Part I: Axiomatic model of the real numbers

- Axioms of an ordered field
- Completeness axiom
- Archimedean principle
- Principle of mathematical induction
- Countable and uncountable sets

Thomson/Bruckner/Bruckner: 1.1–1.10, 2.3

Topics for Test 1

Part II: Sequences and infinite sums

- Limits of sequences
- Bolzano-Weierstrass theorem
- Cauchy sequences
- Convergence of series
- Tests for convergence
- Absolute convergence

Thomson/Bruckner/Bruckner: 2.1–2.2, 2.4–2.13, 3.1–3.2, 3.4–3.7

Axioms of real numbers

Definition. The set $\mathbb{R}$ of real numbers is a set satisfying the following postulates:

Postulate 1. $\mathbb{R}$ is a field.

Postulate 2. There is a strict linear order $<$ on $\mathbb{R}$ that makes it into an ordered field.

Postulate 3 (Completeness Axiom).
If a nonempty subset $E \subset \mathbb{R}$ is bounded above, then E has a supremum.

Theorems to know

Theorem (Archimedean Principle) For any real number $\varepsilon > 0$ there exists a natural number n such that $n\varepsilon > 1$.

Theorem (Principle of mathematical induction) Let $P(n)$ be an assertion depending on a natural variable n. Suppose that

- $P(1)$ holds,
- whenever $P(k)$ holds, so does $P(k+1)$.

Then $P(n)$ holds for all $n \in \mathbb{N}$.

Theorem If $A_1, A_2, \ldots$ are finite or countable sets, then the union $A_1 \cup A_2 \cup \ldots$ is also finite or countable. As a consequence, the sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{N} \times \mathbb{N}$ are countable.

Theorem The set $\mathbb{R}$ is uncountable.

Limit theorems for sequences

Theorem If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$ and $x_n \leq w_n \leq y_n$ for all sufficiently large n, then $\lim_{n\to\infty} w_n = a$.

Theorem If $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$, and $x_n \leq y_n$ for all sufficiently large n, then $a \leq b$.

Theorem If $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$, then $\lim_{n\to\infty}(x_n + y_n) = a + b$, $\lim_{n\to\infty}(x_n - y_n) = a - b$, and $\lim_{n\to\infty} x_n y_n = ab$. If, additionally, $b \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n/y_n = a/b$.

More theorems on sequences

Theorem Any monotonic sequence converges to a limit if bounded, and diverges to $+\infty$ or $-\infty$ otherwise.

Theorem (Bolzano-Weierstrass) Every bounded sequence of real numbers has a convergent subsequence.

Theorem Any Cauchy sequence is convergent.

Tests for convergence of series

- Trivial Test
- Cauchy Criterion
- Direct Comparison Test
- Ratio Test
- Root Test
- Condensation Test
- Integral Test
- Alternating Series Test
- Dirichlet's Test
- Abel's Test

Sample problems for Test 1

Problem 1. Prove the following version of the Archimedean property: for any positive real numbers x and y there exists a natural number n such that $nx > y$.

Problem 2. Prove that for any $n \in \mathbb{N}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Problem 3. Given a set X, let $\mathcal{P}(X)$ denote the set of all subsets of X. Prove that $\mathcal{P}(X)$ is not of the same cardinality as X.

Sample problems for Test 1

Problem 4. Let $x_1 = a > 0$ and $x_{n+1} = 2\sqrt{x_n}$ for all $n \in \mathbb{N}$. Prove that the sequence $\{x_n\}$ is convergent and find its limit.

Problem 5. Suppose $\{r_n\}$ is a sequence that enumerates all rational numbers. Prove that every real number is a limit point of this sequence.

Sample problems for Test 1

Problem 6. For each of the following series, determine whether the series converges and whether it converges absolutely:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$, **(ii)** $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+2^n\cos n}{n!}$,

(iii) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\log n}$.

Problem 1. Prove the following version of the Archimedean property: for any positive real numbers x and y there exists a natural number n such that $nx > y$.

Proof: Let E be the set of all natural numbers n such that $(n-1)x \leq y$. We are going to show that the set E is nonempty and bounded above (so that $\sup E$ exists due to the Completeness Axiom). Observe that $(1-1)x = 0 < y$. Hence $1 \in E$, in particular, E is not empty. Further, if $(n-1)x \leq y$ then $n - 1 \leq yx^{-1}$ and $n \leq 1 + yx^{-1}$. Therefore $1 + yx^{-1}$ is an upper bound for E.

Now we know that $m = \sup E$ is a well-defined real number. Since $\sup E$ is the least upper bound for the set E and $m - 1 < m$, the number $m - 1$ is not an upper bound for E. Hence there exists $n \in E$ such that $n > m - 1$. Then $n + 1 > m$, which implies that $n + 1 \notin E$. At the same time, $n + 1 \in \mathbb{N}$ since $n \in E \subset \mathbb{N}$. Therefore $((n+1)-1)x > y$, that is, $nx > y$.

Problem 2. Prove that for any $n \in \mathbb{N}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Proof: The proof is by induction on n. First we consider the case $n = 1$. In this case the formula reduces to $1^3 = \frac{1^2 \cdot 2^2}{4}$, which is a true equality. Now assume that the formula holds for $n = k$, that is,

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

Adding $(k+1)^3$ to both sides of this equality, we get

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2 \left(\frac{k^2}{4} + (k+1) \right) = (k+1)^2 \, \frac{k^2+4k+4}{4} = \frac{(k+1)^2(k+2)^2}{4},$$

which means that the formula holds for $n = k + 1$ as well. By induction, the formula holds for any natural number n.

Remark. We have proved that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Also, it is known that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

It follows that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

for all $n \in \mathbb{N}$.

Problem 3. Given a set X, let $\mathcal{P}(X)$ denote the set of all subsets of X. Prove that $\mathcal{P}(X)$ is not of the same cardinality as X.

Proof: We have to prove that there is no bijective map of X onto $\mathcal{P}(X)$. Let us consider an arbitrary map $f : X \to \mathcal{P}(X)$. The image $f(x)$ of an element $x \in X$ under this map is a subset of X. We define a set

$$E = \{x \in X \mid x \notin f(x)\}.$$

By definition of the set E, any element $x \in X$ belongs to E if and only if it does not belong to $f(x)$. As a consequence, $E \neq f(x)$ for all $x \in X$. Hence the map f is not onto. In particular, it is not bijective.

Problem 4. Let $x_1 = a > 0$ and $x_{n+1} = 2\sqrt{x_n}$ for all $n \in \mathbb{N}$. Prove that the sequence $\{x_n\}$ is convergent and find its limit.

If $x > 0$ then $2\sqrt{x}$ is well defined and positive. It follows by induction that each x_n, $n \in \mathbb{N}$ is well defined and positive.

Assume $x_n \to L$ as $n \to \infty$. Then $x_{n+1} \to L$ as $n \to \infty$. Since $x_{n+1}^2 = (2\sqrt{x_n})^2 = 4x_n$, the limit theorems imply that $L^2 = 4L$. Hence $L = 0$ or 4.

Suppose that $0 < x_n < 4$ for some $n \in \mathbb{N}$. Then $x_{n+1} = 2\sqrt{x_n} < 2\sqrt{4} = 4$ and $x_{n+1} = 2x_n/\sqrt{x_n} > 2x_n/\sqrt{4} = x_n$. Similarly, if $x_n > 4$ then $x_{n+1} = 2\sqrt{x_n} > 2\sqrt{4} = 4$ and $x_{n+1} = 2x_n/\sqrt{x_n} < 2x_n/\sqrt{4} = x_n$. Finally, if $x_n = 4$ then $x_{n+1} = 2\sqrt{x_n} = 2\sqrt{4} = 4$. In each of the three cases, it follows by induction that the sequence $\{x_n\}$ is monotonic and bounded. Hence it is convergent. Moreover, the sequence is bounded below by $a > 0$ if it is increasing and by 4 otherwise. Thus the limit cannot be 0.

Problem 5. Suppose $\{r_n\}$ is a sequence that enumerates all rational numbers. Prove that every real number is a limit point of this sequence.

Proof: Let α be an arbitrary real number. We need to show that the sequence $\{r_n\}$ has a subsequence converging to α. Recall that every interval $(a, b) \subset \mathbb{R}$ contains a rational number. In particular, for any $n \in \mathbb{N}$ there is an index k_n such that $r_{k_n} \in (\alpha, \alpha + 1/n)$. Then $|r_{k_n} - \alpha| < 1/n$, which implies that $r_{k_n} \to \alpha$ as $n \to \infty$.

The sequence $\{r_{k_n}\}$ is not necessarily a subsequence of $\{r_n\}$ as the sequence of indices $\{k_n\}$ need not be increasing. However any rational number r can occur in it only finitely many times (since inequalities $\alpha < r < \alpha + 1/n$ cannot hold for arbitrarily large n). It follows that the sequence of indices has an increasing subsequence $\{k_{n_m}\}$. Then the sequence $\{r_{k_{n_m}}\}$ is both a subsequence of $\{r_{k_n}\}$ (and hence convergent to α) and a subsequence of $\{r_n\}$.

Problem 6. For each of the following series, determine if the series converges and if it converges absolutely:

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$, **(ii)** $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+2^n\cos n}{n!}$, **(iii)** $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\log n}$.

The first series diverges since

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{\left(\sqrt{n+1}+\sqrt{n}\right)^2}>\sum_{n=1}^{\infty}\frac{1}{4(n+1)}=+\infty.$$

The second series can be represented as $\sum_{n=1}^{\infty}(b_n+c_n\cos n)$, where $b_n=\sqrt{n}/n!$ and $c_n=2^n/n!$ for all $n\in\mathbb{N}$. The series $\sum_{n=1}^{\infty}b_n$ and $\sum_{n=1}^{\infty}c_n$ both converge (due to the Ratio Test), and so does $\sum_{n=1}^{\infty}(b_n+c_n)$. Since $|b_n+c_n\cos n|\le b_n+c_n$ for all $n\in\mathbb{N}$, the series $\sum_{n=1}^{\infty}(b_n+c_n\cos n)$ converges absolutely due to the Direct Comparison Test.

Finally, the third series converges (due to the Alternating Series Test), but not absolutely (due to the Integral Test).