MATH 409

Advanced Calculus I

Lecture 19b: Topology of the real line: classification of points.

Classification of points

Let $E \subset \mathbb{R}$ be a subset of the real line and $x \in \mathbb{R}$ be a point. Recall that for any $\varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon)$ is called the ε-**neighborhood** of the point x as it consists of all points at distance less than ε from x.

Definition. The point x is called an **interior point** of the set E if for some $\varepsilon > 0$ the entire ε-neighborhood $(x - \varepsilon, x + \varepsilon)$ is contained in E. The point x is called an **exterior point** of E if for some $\varepsilon > 0$ the ε-neighborhood $(x - \varepsilon, x + \varepsilon)$ is disjoint from E. The point x is called a **boundary point** of E if for any $\varepsilon > 0$ the ε-neighborhood $(x - \varepsilon, x + \varepsilon)$ contains both a point in E and another point not in E.

Remark. Every interior point of the set E must belong to E. Every exterior point of E must not belong to E. Any particular boundary point may or may not be in E.

Examples

- $E = (a, b)$, an open interval.

The interior points are points in (a, b). The exterior points are points in $(-\infty, a) \cup (b, +\infty)$. The boundary points are a and b.

- $E = [a, b]$, a closed interval.

The interior points are points in (a, b). The exterior points are points in $(-\infty, a) \cup (b, +\infty)$. The boundary points are a and b.

- $E = [0, 1) \cup [2, 3)$.

The interior points are points in $(0, 1) \cup (2, 3)$. The exterior points are points in $(-\infty, 0) \cup (1, 2) \cup (3, +\infty)$. The boundary points are 0, 1, 2 and 3.

Examples

- $E = \mathbb{R}$, the entire real line.

Every point is interior. There are no boundary or exterior points.

- $E = \emptyset$, the empty set.

Every point is exterior. There are no interior or boundary points.

- $E = \mathbb{Q}$, the set of rational numbers.

Every open interval (a, b) contains a rational number (the rational numbers are dense). Also, (a, b) contains an irrational number (since $\mathbb{Q}$ is countable while the interval is not). Therefore every point of $\mathbb{R}$ is a boundary point for $\mathbb{Q}$. There are no interior or exterior points.

Examples

- $E = \mathbb{N}$, the natural numbers.

Every natural number is a boundary point. Any non-natural number is an exterior point. There are no interior points.

- $E = \{1, 1/2, 1/3, 1/4, \ldots\}$.

The boundary points are all points of E and 0. All the other point are exterior. There are no interior points.