MATH 409

Advanced Calculus I

Lecture 20: Topology of the real line: open and closed sets.

Classification of points

Let $E \subset \mathbb{R}$ be a subset of the real line and $x \in \mathbb{R}$ be a point. Recall that for any $\varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon)$ is called the ε-**neighborhood** of the point x as it consists of all points at distance less than ε from x.

Definition. The point x is called an **interior point** of the set E if for some $\varepsilon > 0$ the entire ε-neighborhood $(x - \varepsilon, x + \varepsilon)$ is contained in E. The point x is called an **exterior point** of E if for some $\varepsilon > 0$ the ε-neighborhood $(x - \varepsilon, x + \varepsilon)$ is disjoint from E. The point x is called a **boundary point** of E if for any $\varepsilon > 0$ the ε-neighborhood $(x - \varepsilon, x + \varepsilon)$ contains both a point in E and another point not in E.

Remark. Every interior point of the set E must belong to E. Every exterior point of E must not belong to E. Any particular boundary point may or may not be in E.

Examples

- $E = (a, b)$, an open interval.

The interior points are points in (a, b). The exterior points are points in $(-\infty, a) \cup (b, +\infty)$. The boundary points are a and b.

- $E = [a, b]$, a closed interval.

The interior points are points in (a, b). The exterior points are points in $(-\infty, a) \cup (b, +\infty)$. The boundary points are a and b.

- $E = [0, 1) \cup [2, 3)$.

The interior points are points in $(0, 1) \cup (2, 3)$. The exterior points are points in $(-\infty, 0) \cup (1, 2) \cup (3, +\infty)$. The boundary points are 0, 1, 2 and 3.

Examples

- $E = \mathbb{R}$, the entire real line.

Every point is interior. There are no boundary or exterior points.

- $E = \emptyset$, the empty set.

Every point is exterior. There are no interior or boundary points.

- $E = \mathbb{Q}$, the rational numbers.

Every open interval (a, b) contains a rational number (the rational numbers are dense). Also, (a, b) contains an irrational number (since $\mathbb{Q}$ is countable while the interval is not). Therefore every point of $\mathbb{R}$ is a boundary point for $\mathbb{Q}$. There are no interior or exterior points.

Examples

- $E = \mathbb{N}$, the natural numbers.

Every natural number is a boundary point. Any non-natural number is an exterior point since the complement $\mathbb{R} \setminus \mathbb{N}$ is a union of open intervals: $\mathbb{R} \setminus \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \ldots$
There are no interior points.

- $E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$, a monotonic sequence.

The boundary points are all points of E and 0 (the limit of the sequence). All the other points are exterior. There are no interior points.

Let $E \subset \mathbb{R}$ be a subset of the real line.

Definition. The set of all interior points of E is called the **interior** of E and denoted $\mathrm{int}(E)$. The set of all boundary points of E is called the **boundary** of E and denoted ∂E. The set of all exterior points of E is called the **exterior** of E.

Proposition 1 The exterior of the set E coincides with $\mathrm{int}(\mathbb{R} \setminus E)$, the interior of its complement.

Proposition 2 The boundary of the set E coincides with the boundary of its complement: $\partial E = \partial(\mathbb{R} \setminus E)$.

Proposition 3 The real line $\mathbb{R}$ is the disjoint union of three sets: $\mathbb{R} = \mathrm{int}(E) \cup \partial E \cup \mathrm{int}(\mathbb{R} \setminus E)$.

Proposition 4 $\mathrm{int}(E) \subset E \subset \mathrm{int}(E) \cup \partial E$.

Limit points of a set

Let $E \subset \mathbb{R}$ be a subset of the real line.

Definition. A point $x \in \mathbb{R}$ is called a **limit point** of the set E if there exists a sequence $x_1, x_2, x_3, \dots$ such that each x_n belongs to E and $x_n \to x$ as $n \to \infty$.

Remark. Elements of the sequence $\{x_n\}$ need not be distinct. In particular, every point $x \in E$ is a limit point of E, as the limit of a constant sequence $x, x, x, \dots$

Theorem A point $x \in \mathbb{R}$ is a limit point of a set $E \subset \mathbb{R}$ if and only if for any $\varepsilon > 0$ the ε-neighborhood $(x - \varepsilon, x + \varepsilon)$ contains at least one element of E.

Corollary For any set $E \subset \mathbb{R}$, the set of all limit points of E is $\mathrm{int}(E) \cup \partial E$.

Another classification of points

Let $E \subset \mathbb{R}$ be a subset of the real line and $x \in \mathbb{R}$ be a point.

Definition. The point x is called an **accumulation point** of the set E if for any $\varepsilon > 0$ the ε-neighborhood $(x - \varepsilon, x + \varepsilon)$ contains infinitely many elements of E. The point x is called an **isolated point** of E if $(x - \varepsilon, x + \varepsilon) \cap E = \{x\}$ for some $\varepsilon > 0$.

Theorem A point $x \in \mathbb{R}$ is an accumulation point of a set $E \subset \mathbb{R}$ if and only if for any $\varepsilon > 0$ the ε-neighborhood $(x - \varepsilon, x + \varepsilon)$ contains at least one element of E different from x.

Corollary For any set $E \subset \mathbb{R}$, the set of all limit points of E is a disjoint union of the set of its accumulation points and the set of its isolated points.

Examples

- $E = \mathbb{N}$, the natural numbers.

Every natural number is an isolated point. There are no accumulation points.

- $E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$, a monotonic sequence.

Every element of E is an isolated point. The only accumulation point is 0.

- $E = \mathbb{Q}$, the rational numbers.

Every point of $\mathbb{R}$ is an accumulation point for $\mathbb{Q}$. There are no isolated points.

Open and closed sets

Definition. A subset $E \subset \mathbb{R}$ of the real line is called **open** if every point of E is an interior point. The subset E is called **closed** if it contains all of its limit points (or, equivalently, if it contains all of its boundary points).

Properties of open and closed sets.

- Any open interval (a, b) is an open set.
- Any closed interval $[a, b]$ is a closed set.
- If a set E is open then the complement $\mathbb{R} \setminus E$ is closed.
- If a set E is closed then the complement $\mathbb{R} \setminus E$ is open.
- The empty set and the entire real line $\mathbb{R}$ are both closed and open (in fact, these are the only sets with this property).
- Intersection of two open sets is also open.
- Union of any collection of open sets is also open.
- Union of two closed sets is also closed.
- Intersection of any collection of closed sets is also closed.